MATRIX

TYPES OF SYMMETRIES*

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1. Symmetric and skew-symmetric matrices. It is well known that a matrix $A=(A_{ij})$ can be written as the sum of a symmetric and of a skew-symmetric matrix: $A=S+Q$, where

$$S = \tfrac{1}{2}(A + A')$$

is symmetric, and

$$Q = \tfrac{1}{2}(A - A')$$

is skew-symmetric. Here A' denotes the transpose of A. In index notation $A_{ij}=S_{ij}+Q_{ij}$, where

$$S_{ij} = \tfrac{1}{2}(A_{ij} + A_{ji}) \quad \text{and} \quad Q_{ij} = \tfrac{1}{2}(A_{ij} - A_{ji}). \tag{1}$$

The matrix A_{ij} is symmetric if $Q_{ij}=0$, and skew-symmetric if $S_{ij}=0$. It is significant that a symmetric matrix will transform under a linear transformation into a symmetric matrix, and a skew-symmetric matrix will transform into a skew-symmetric matrix.

2. Tensors. In the consideration of types of symmetries for ordered arrays which are invariant under general linear transformations, it is convenient to use the viewpoint and terminology of tensor algebra. If the coordinate transformation is

$$\bar{x}^k = T_i^k x^i \qquad (i = 1, 2, \cdots, n),$$

a 2-way matrix A_{ij} is a covariant tensor of order two if its components satisfy

$$A_{ij} = |T|^w T_i^k T_j^l \overline{A}_{kl},$$

for some constant w, where $|T|$ is the determinant of the transformation matrix T_i^k. The fact that the types of symmetries discussed in Section 1 are invariant under a general linear transformation may be expressed by saying that these symmetries have tensorial significance. Furthermore, for a 2-indexed tensor they are the only types of symmetries which do have tensorial significance.

More generally, if an ordered array of scalars, or point functions, $A_{i_1 i_2 \cdots i_p}$ transforms according to the law

$$A_{i_1 i_2 \cdots i_p} = |T|^w T_{i_1}^{k_1} T_{i_2}^{k_2} \cdots T_{i_p}^{k_p} \overline{A}_{k_1 k_2 \cdots k_p},$$

when the variable coordinate transformation is $\bar{x}^k = T_i^k x^i$, then $A_{i_1 i_2 \cdots i_p}$ is a covariant tensor of order p. In this paper we shall discuss general types of symmetries, of which complete symmetry and complete skew-symmetry are special cases, and shall consider some applications of these general types of symmetries

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to tensors of order greater than two. These general classes of symmetries were first explored by A. Young [1], and later studied by J. A. Schouten [2], by J. von Neumann [3], by H. Weyl [4], [5], and more recently by the present writers [6], [7].

3. Symmetries for third order tensors. From an arbitrary 3-indexed tensor A_{ijk} may be obtained a completely symmetric tensor

$$S_{ijk} = \frac{1}{3!}[A_{ijk} + A_{ikj} + A_{jki} + A_{jik} + A_{kij} + A_{kji}], \tag{2}$$

and a completely skew-symmetric tensor

$$Q_{ijk} = \frac{1}{3!}[A_{ijk} - A_{ikj} + A_{jki} - A_{jik} + A_{kij} - A_{kji}]. \tag{3}$$

However

$$R_{ijk} \equiv A_{ijk} - (S_{ijk} + Q_{ijk}) \tag{4}$$

is not the zero tensor. In fact

$$R_{ijk} = \frac{2}{3!}[2A_{ijk} - A_{jki} - A_{kij}]. \tag{5}$$

The tensor R_{ijk} possesses no very obvious type of symmetry; however

$$R_{ijk} = S^{(1)}_{ijk} + S^{(2)}_{ijk}, \tag{6}$$

where

$$\begin{aligned} S^{(1)}_{ijk} &= \frac{2}{3!}\{(A_{ijk} - A_{kji}) + (A_{jik} - A_{kij})\}; \\ S^{(2)}_{ijk} &= \frac{2}{3!}\{(A_{ijk} - A_{jik}) + (A_{kji} - A_{jki})\}. \end{aligned} \tag{7}$$

The tensor $S^{(1)}_{ijk}$ is symmetric in i and j, and the tensor $S^{(2)}_{ijk}$ is symmetric in i and k. Both of these tensors clearly would vanish if A_{ijk} were completely symmetric or completely skew-symmetric. Similarly

$$R_{ijk} = Q^{(1)}_{ijk} + Q^{(2)}_{ijk} \tag{8}$$

where

$$\begin{aligned} Q^{(1)}_{ijk} &= \frac{2}{3!}\{(A_{ijk} + A_{jik}) - (A_{kji} + A_{jki})\}; \\ Q^{(2)}_{ijk} &= \frac{2}{3!}\{(A_{ijk} + A_{kji}) - (A_{jik} + A_{kij})\}. \end{aligned} \tag{9}$$

The tensor $Q^{(1)}_{ijk}$ is skew-symmetric in i and k, and the tensor $Q^{(2)}_{ijk}$ is skew-symmetric in i and j, and each of these tensors would vanish if A_{ijk} were completely symmetric or completely skew-symmetric. The tensor R_{ijk} can be written in other ways as the sum of two "partially symmetric" tensors, or of two "partially skew-symmetric" tensors; but the two relations (6) and (8) suffice to show that R_{ijk} has its peculiar type of symmetry different from the complete symmetry of S_{ijk} and different from the complete skew-symmetry of Q_{ijk}.

Analogous to (1) we may write

$$A_{ijk} = S_{ijk} + R_{ijk} + Q_{ijk}. \tag{10}$$

Thus we see that for an arbitrary 3-index tensor to be symmetric it is necessary that the skew-symmetric part Q_{ijk} be zero, but this condition is not sufficient; in addition the part we have designated by R_{ijk} must be zero.

A 4-indexed tensor may be written as a sum of five tensors, corresponding to (10), and each one of these, except for the completely symmetric and completely skew-symmetric tensors, may be written as the sum of several tensors with partial skew-symmetries (symmetries) of different natures. How this is done for a 4-indexed tensor or, more generally, for a p-indexed tensor will be evident from a discussion of A. Young's symmetry operators.

4. Young's symmetry operators. With p letters we may construct the tableau

$$[\alpha] = [\alpha_1, \alpha_2, \cdots, \alpha_h]: \quad \begin{matrix} a_{11} & a_{12} \cdots a_{1\alpha_1} \\ a_{21} & a_{22} \cdots a_{2\alpha_2} \\ \cdots & \cdots \\ a_{h1} & a_{h2} \cdots a_{h\alpha_h} \end{matrix}$$

where $\alpha_i \geqq \alpha_{i+1} > 0$ for all i, and $\sum_{i=1}^{h} \alpha_i = p$; the letters a_{ij} stand for the numbers 1 to p in some order. There is a tableau for each partition of p; the tableau and the partition, and in fact the corresponding irreducible representation of the symmetric group on p letters, are represented by the same symbol $[\alpha]$. In this connection it is convenient to use an exponential notation when two or more adjacent α's are equal. Thus $[2, 1^2]$ denotes the partition $[2, 1, 1]$ of 4.

From the ith row of $[\alpha]$ we can construct the substitutional operator $\{a_{i1}a_{i2} \cdots a_{i\alpha_i}\}$, which represents the sum of all permutations of the symmetric group on the α_i letters exhibited between the brackets; the product of these h operators we denote by P_α. Similarly from the jth column of $[\alpha]$ we construct the operator $\{a_{1j}a_{2j} \cdots\}'$, which represents the sum of all the even permutations minus the sum of all the odd permutations of the symmetric group on the letters standing in the jth column; the product of these column operators we denote by N_α. With any arrangement of the p letters in the tableau form $[\alpha]$ we may associate the product $P_\alpha N_\alpha$ or the product $N_\alpha P_\alpha$.

With the partition $[\alpha] = [2, 1]$ of 3 we may construct the tableau form $\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}$ and the 3! particular tableaux

$$(11)\qquad \begin{matrix}(1)\\12\\3\end{matrix}\qquad \begin{matrix}(2)\\13\\2\end{matrix}\qquad \begin{matrix}(3)\\21\\3\end{matrix}\qquad \begin{matrix}(4)\\23\\1\end{matrix}\qquad \begin{matrix}(5)\\32\\1\end{matrix}\qquad \begin{matrix}(6)\\31.\\2\end{matrix}$$

Let $P_t \equiv P_\alpha^{(t)}$ and $N_t \equiv N_\alpha^{(t)}$ be associated with the tableau numbered (t). Then $P_1 = \{12\}\{3\} = \{12\} = 1 + (12)$; $N_1 = \{13\}'\{2\}' = \{13\}' = 1 - (13)$; and

$$P_1N_1 = [1 + (12)][1 - (13)] = 1 + (12) - (13) - (123).$$

Here we use the parentheses ( ) in the conventional manner of circular permutations. Using P_1N_1 as a substitutional operator on the indices of A_{ijk} we obtain

$$\begin{aligned} A_{ijk} \cdot P_1N_1 &= A_{ijk} \cdot 1 + A_{ijk}(12) - A_{ijk}(13) - A_{ijk}(123) \\ &= A_{ijk} + A_{jik} - A_{kji} - A_{jki}. \end{aligned}$$

We note that

$$(12)\qquad {}_{[\alpha]}A_{ijk}^{(1)} \equiv A_{ijk} \cdot \frac{2}{3!} P_1N_1 = \frac{2}{3!}\left[(A_{ijk} + A_{jik}) - (A_{kji} + A_{jki})\right]$$

is the tensor $Q_{ijk}^{(1)}$ of (9). Similarly

$$P_2N_2 = [1 + (13)][1 - (12)] = 1 + (13) - (12) - (132)$$

and

$$(13)\qquad {}_{[\alpha]}A_{ijk}^{(2)} \equiv A_{ijk} \cdot \frac{2}{3!} P_2N_2 = \frac{2}{3!}\left[(A_{ijk} + A_{kji}) - (A_{jik} + A_{kij})\right]$$

is the tensor $Q_{ijk}^{(2)}$ of (9).

Reversing the order of P_1 and N_1 we get

$$N_1P_1 = [1 - (13)][1 + (12)] = 1 + (12) - (13) - (132).$$

The corresponding tensor $A_{ijk} \cdot \frac{2}{3!} N_1P_1$ is the tensor $S_{ijk}^{(1)}$ of (7); likewise $A_{ijk} \cdot \frac{2}{3!} N_2P_2$ is the tensor $S_{ijk}^{(2)}$ of (7).

Note that the tensor $A_{ijk} \cdot \frac{2}{3!} P_1N_1$ is skew-symmetric in i, k, its first and third indices; this is a consequence of the fact that we operated last with N_1. Similarly the tensor $A_{ijk} \cdot \frac{2}{3!} N_1P_1$ is symmetric in i, j, its first and third indices, since we operated last with P_1. This illustrates the following fact: *The tensor obtained by operating on $A_{i_1i_2\cdots i_p}$ with the operator $P_\alpha^{(t)}N_\alpha^{(t)}$ for a given tableau is skew-symmetric in each set of indices belonging to the same column of the tableau, and the tensor obtained by operating on $A_{i_1i_2\cdots i_p}$ with the corresponding $N_\alpha^{(t)}P_\alpha^{(t)}$ is symmetric in each set of indices belonging to the same row of the tableau.*

Taking the sum of the 3! products P_tN_t corresponding to the tableau (11) (or of the 3! products N_tP_t) we obtain

$$(14)\qquad 3[2 - (123) - (132)]$$

which, when used as an operator on A_{ijk} yields the tensor R_{ijk} of (5) except for

a constant factor. But the operator (14) may also be obtained, aside from the factor 3, by taking the sum of P_1N_1 and P_2N_2 (or the sum of N_1P_1 and N_2P_2). In general we write

$$T_\alpha = \left(\frac{f_\alpha}{p!}\right)^2 \sum_{i=1}^{p!} P_\alpha^{(i)} N_\alpha^{(i)},$$

where the constant f_α is given by

$$f_\alpha = p! \frac{\prod_{r<s}^{h} (l_r - l_s)}{\prod_{r=1}^{h} l_r!}, \qquad l_r = \alpha_r + h - r (r = 1, \cdots, h). \tag{15}$$

For $p=2, f_{[2]}=1; f_{[1^2]}=1$. For $p=3, f_{[3]}=1; f_{[2,1]}=2, f_{[1^3]}=1$. For $p=4, f_{[4]}=1; f_{[3,1]}=3; f_{[2^2]}=2; f_{[2,1^2]}=3; f_{[1^4]}=1$. Alternatively, f_α is the degree of the irreducible representation $[\alpha]$, and this may be obtained as the characteristic of (1^p) from the character table of the symmetric group on p letters [8].

A tableau $[\alpha]$ is defined to be standard if the letters in each row and column appear in the order of some given sequence. Thus of the tableaux (11) those designated as (1) and (2) may be taken as standard. It can be shown that exactly f_α of the $p!$ tableaux are standard. Moreover, it can be shown that T_α may be expressed in terms of the standard tableaux, this expression being

$$T_\alpha = \frac{f_\alpha}{p!} [P_1N_1M_1 + P_2N_2M_2 + \cdots + P_fN_fM_f]. \tag{16}$$

The permutation operators M_t are introduced so that the f_α "prepared" operators $P_tN_tM_t$ are orthogonal. For all cases for $p<5$ the M's are each the identity permutation. In some cases, as for example when $p=5$, $[\alpha]=[3, 2]$ and $[\alpha]=[2^2, 1]$, some of the M's are not the identity and must be determined [9].

5. Symmetries for fourth order tensors. As we have said above, a 4-indexed tensor $A_{i_1i_2i_3i_4}$ may be written uniquely as the sum of five tensors corresponding to the five partitions of the number 4. For $[\alpha]=[2^2], f_\alpha=2$, and we may take as our two standard forms $\frac{12}{34}$ and $\frac{13}{24}$. One may obtain ${}_{[2^2]}A_{i_1i_2i_3i_4}$ from $A_{i_1i_2i_3i_4}$ by applying to the latter the operator

$$\frac{2}{4!} [\{12\}\{34\}\{13\}'\{24\}' + \{13\}\{24\}\{12\}'\{34\}']. \tag{17}$$

This gives

$${}_{[2^2]}A_{i_1i_2i_3i_4} = \frac{2}{4!} [2A^{(1^4)}_{i_1i_2i_3i_4} + 0 \cdot A^{(1^2,2)}_{i_1i_2i_3i_4} - 1 \cdot A^{(1,3)}_{i_1i_2i_3i_4} + 0 \cdot A^{(4)}_{i_1i_2i_3i_4} + 2 \cdot A^{(2^2)}_{i_1i_2i_3i_4}], \tag{18}$$

where $A^{(\rho)}_{i_1i_2i_3i_4}$ is the sum of all tensors obtained from $A_{i_1i_2i_3i_4}$ by subjecting the

indices to the different permutations of class (ρ). The coefficient of $A^{(\rho)}_{i_1i_2i_3i_4}$ within the brackets is the characteristic for class (ρ) of the irreducible representation $[2^2]$. (See paper [6].) Evidently we could have avoided the use of Young's diagrams, and worked with the character table instead, provided we were interested only in the tensor corresponding to the Young tableau form. However, a tensor associated with a particular Young diagram is perhaps best considered in relation to the corresponding Young operator.

6. Number of independent components. The number of independent (scalar) components of $A_{i_1i_2\cdots i_p}$ is n^p, n being the dimension of the coordinate system. If c_α denotes the number of components of ${}_{[\alpha]}A_{i_1i_2\cdots i_p}$, then

$$n^p = c_{[p]} + \cdots + c_{[\alpha]} + \cdots + c_{[1^p]}. \tag{19}$$

For $p=2$ the relation connecting the dimensionalities of the special symmetric tensors is

$$n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2},$$

and for $p=3$ the relation is

$$n^3 = \frac{n(n+1)(n+2)}{3!} + \frac{4n(n^2-1)}{3!} + \frac{n(n-1)(n-2)}{3!}.$$

Schouten [2] has obtained expressions for the c_α's in terms of n for $p=4$, but the difficulties of his method become great for larger values of p. From the definition of the rank r_α, of the immanent tensor ${}_{[\alpha]}I^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_p}$ introduced in [6] as given in Section 4 of [10], it has been shown [7] that $c_\alpha = r_\alpha$. Since r_α may be read off from the character table for the symmetric group on p letters, it can be said that expressions for c_α are readily obtainable for those values of p for which the character tables are available, these being for $p \leqq 13$, [11], [12], [13].

Knowing the number of independent scalar components, c_α, of a tensor ${}_{[\alpha]}A_{i_1i_2\cdots i_p}$ corresponding to a Young T_α, we may obtain the number of independent scalar components, ${}_1c_\alpha$, in the tensor corresponding to a particular Young diagram. For as a consequence of Theorems VI and VIII of [10] we have

$${}_1c_\alpha = \frac{1}{f_\alpha} \cdot r_\alpha. \tag{20}$$

The number of independent scalar components [7] in the tensor ${}_{[2^1]}A_{i_1i_2i_3i_4}$ is

$$c_{[2^2]} = n\binom{n+1}{3},$$

and the number of such components in the tensor corresponding to the particular Young diagram $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$ is

$${}_1c_{[2^2]} = \frac{n}{2}\binom{n+1}{3}.$$

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BOOLEAN REPRESENTATION THEORY

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1. Introduction. Recent proofs of Stone's theorem on representations of Boolean algebras [6], [7], have been given by Frink [3] and McCoy-Montgomery [4], [5]. The simplest form of the theorem asserts that every Boolean algebra (or more generally Boolean ring) is isomorphic to a Boolean algebra (or Boolean ring) of sets or classes. The latter is called the representation of the former. Frink's proof made use of Zorn's Lemma [8] and a new set of postulates for Boolean algebras. McCoy and Montgomery used a principle formulated by them concerning the direct product of rings. Stone's original proof [7] made use of detailed transfinite induction and rather specialized algebraic arguments.

It is proposed here to give a composite proof, using methods due to all of the above authors, and providing what seems to be a simplified treatment of the main theory. At the same time, the proof will depend on standard algebraic methods in the theory of ideals. The proof to be outlined follows the general plan of Stone's proof but the crucial arguments are not so specialized as his, nor is the proof as general as that of McCoy and Montgomery. We follow Frink in using Zorn's Lemma. No consideration will be given here to the approach to the representation problem from the standpoint of lattice theory.*

* See Garrett Birkhoff, Lattice Theory, American Mathematical Society, Colloquium Publications, Vol. 25, 1940, Chapters V, VI.

2. Properties of Boolean rings. First, we recall a few elementary facts about Boolean rings as proved by Stone [7]. A Boolean ring is defined as a ring in which every element, a, is idempotent, $aa=a$. It follows that every element is its own additive inverse ($a+a=0$), and the ring is commutative. A Boolean ring with unit element e (such that $ae=a$ for every element a) becomes a Boolean algebra when Boolean addition ($\vee$) is defined by the relation

$$a \vee b = (a + b) + ab$$

and the Boolean complement a' by

$$a' = a + e.$$

It follows that $a+a'=e$, $aa'=0$. A non-trivial Boolean ring (one containing more than one element) is a field if and only if it has two elements, in which case it is abstractly identical with the field F_2 (of integers 0, 1, mod 2).* (A Boolean ring with more than two elements always has divisors of zero.) Every Boolean ring can be imbedded in a Boolean ring with unit element. Hence in order to prove that a representation exists for an arbitrary Boolean ring it is sufficient to prove the same fact for any Boolean ring with unit element. This can be done in the latter case by means of sets of homomorphisms of the ring into F_2.

3. Maximal ideals in a Boolean ring. Hereafter, we use B to denote an arbitrary non-trivial Boolean ring with unit element.

Let a be any element $\neq 0$ belonging to B, and let $J(a)$ be the collection of all ideals in B which fail to contain a (a non-empty collection since the ideal 0 belongs to $J(a)$). Consider any subsystem of ideals of $J(a)$ linearly ordered under set inclusion. The union of all ideals of the subsystem is an ideal belonging to $J(a)$. Zorn's principle† asserts that under these circumstances there exists in the collection $J(a)$ at least one ideal J_m which is maximal in $J(a)$. Thus we have:

THEOREM 1. *For any element $a\neq 0$, belonging to B, there exists in B at least one ideal J_m which is maximal with respect to the property of not containing a.*

We next prove that J_m actually is a maximal ideal in the ring, in the sense that any more inclusive ideal is the entire ring.

THEOREM 2. *The ideal J_m of Theorem 1 is a maximal ideal in the ring B.*

Proof. Let K be any ideal in B such that K contains J_m as a proper sub-ideal. J_m is maximal with respect to the property of not containing a; hence K must

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* The operation tables for F_2 are, of course:

$$0+1=1+0=1,\quad 0+0=1+1=0,$$
$$0\cdot 1=1\cdot 0=0\cdot 0=0,\quad 1\cdot 1=1.$$

† The principal provides a short-cut method of transfinite induction. It may be stated as follows: If a collection of sets $[S]$ contains the union of every linearly ordered subsystem of $[S]$ (under set inclusion), then there exists at least one set S_m belonging to $[S]$ which is not a proper subset of any other set in the collection, or, briefly, which is *maximal* in $[S]$.

contain a. The complementary element a' either (1) belongs to J_m, or (2) does not belong to J_m.

In the first case, we know at once that $e(=a+a')$ belongs to K. Thus K is the entire ring B, and J_m is a maximal ideal in B.

In the second case, consider the set C of elements of the form $J_m+a'x$ for all x belonging to B. C is an ideal which contains J_m as a proper sub-ideal. Hence, a, as well as a', belongs to C. Thus e belongs to C and we can write

$$e = j + a'y \begin{Bmatrix} \text{for some element } j \text{ of } J_m \\ \text{and some element } y \text{ of } B \end{Bmatrix}.$$

Therefore $$ae = aj + a(a'y) = aj + 0,$$

or $$a = aj.$$

But since j belongs to ideal J_m (which is closed under multiplication by all elements of B), this means that a also must belong to J_m, which is a contradiction.

Thus case (2) is impossible and the conclusion holds that J_m is maximal in B.

COROLLARY. *For any element $a\neq 0$, belonging to B, there exists at least one ideal, J_m, not containing a, which is maximal in B.*

4. The representation. We now refer to a well-known theorem of ideal theory (see Albert [1], p. 255, or Birkhoff-MacLane [2], p. 356) which may be paraphrased as follows: *Given a commutative ring, R, with unit element, and an ideal, J, in R; then the quotient ring R/J is a field if (and only if) J is a maximal ideal in R.* Thus for any element $a\neq 0$, in the Boolean ring, B, there exists a quotient ring, B/J_m, which is a field. But this quotient ring is automatically a Boolean ring so the field must be abstractly equivalent to the field F_2. Hence we can construct a homomorphism, h, of B into F_2 with the elements of J_m mapped on 0 and the complement of J_m, including a and e, mapped on 1.

We can now restate the corollary of the preceding section as follows:

THEOREM 3. *For each element, $a\neq 0$, belonging to B, there exists at least one homomorphism, h, of B into F_2 such that $h(a)$, the image of a in F_2, $=1$.*

COROLLARY. *There fails to exist a homomorphism, h, of B into F_2 such that $h(x)=1$ if and only if $x=0$ in B.*

The desired representation can now be constructed by means of the following two theorems concerning sets of homomorphisms. The operations $\cap$, $\triangle$, are used to designate intersection and symmetric difference* of sets, respectively. The elements a, b, are now any elements of the ring B, not excluding 0.

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* The intersection of two sets is the set of all elements common to both. The symmetric difference of two sets is the set of all elements belonging to one or the other of the sets but not both.

THEOREM 4. *Let $H(a)$ be the set of all homomorphisms, h, of B into F_2 such that $h(a)=1$.*

Then (1) $H(ab) = H(a) \cap H(b)$,
(2) $H(a+b) = H(a)\ \Delta H(b)$,
(3) $H(a) = H(b)$ implies $a=b$.

Proof. 1. $H(ab) = [\text{set of all } h \text{ such that } h(ab)=1]$. But $h(ab)=h(a)\cdot h(b)$ by definition of homomorphism and inasmuch as $h(a)$, $h(b)$ are both elements of F_2 we have

$$h(a)\cdot h(b)=1 \text{ only if } \textit{both } h(a)=1 \textit{ and } h(b)=1.$$

That is, $H(ab)=H(a)\cap H(b)$.

2. $H(a+b)=[\text{set of all } h \text{ such that } h(a+b)=1]$. Again $h(a+b)=h(a)+h(b)$ and

$$h(a)+h(b)=1 \text{ only if } \textit{either } h(a)=1 \textit{ or } h(b)=1, \text{ but } \textit{not both.}$$

Thus $H(a+b)=H(a)\ \Delta H(b)$.

3. $H(a)=H(b)$ implies that the set $H(a)\ \Delta H(b)$ is empty. In this case, then, $H(a+b)$ is an empty set. But, by the corollary to Theorem 3, $H(x)$ is empty only if $x=0$ in B. Thus $a+b=0$ and $a=b$ (by the Boolean ring property that every element is its own additive inverse).

THEOREM 5. *Under the set operations of $\cap$ and Δ the totality of sets $H(x)$, where x runs through all elements of B, form a Boolean ring. This ring of sets is isomorphic to B under the correspondence $x \to H(x)$, $\times \to \cap$, $+ \to \Delta$.*

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AN ELECTRICAL CHINESE RING PUZZLE

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1. Mechanical Chinese rings. Many of the readers of this magazine are undoubtedly familiar with the mechanical puzzle called Chinese rings. As usually displayed in the novelty shops, it consists of an elongated metal loop intertwined with a series of rings. This is shown diagrammatically in Figure 1. The puzzle consists in removing the loop from the rings. To accomplish this, the rings are dropped or lifted through the loop in a special sequence. Each time a ring is dropped or raised is called a move. The number of moves to get the loop

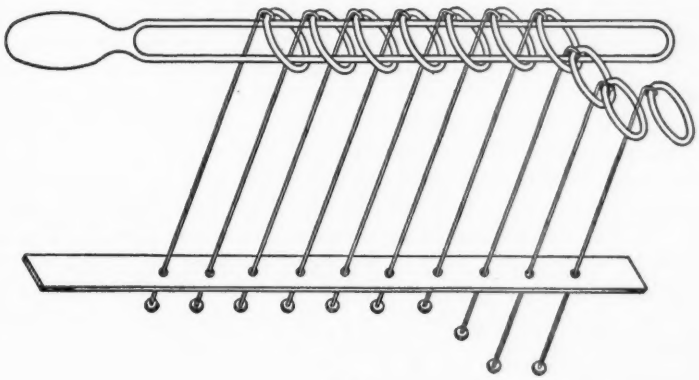

FIG. 1

clear of the rings is obviously a function of the number of rings of which the puzzle is constructed. A 7 ring puzzle requires 85 moves, a 5 ringer only 21, while a 15-ringer requires 21,845 moves.

Unlike many mechanical puzzles, the solution of this one does not depend on cut-and-try methods only. W. W. R. Ball in his "Mathematical Recreations and Essays" and numerous other writers on the subject have expounded the mathematical rules governing this puzzle. The analysis, therefore, need not be given here. The general formula for M, the number of moves to free the loop, in terms of n, the number of rings, is:

$$M = \frac{2^{n+1} - 1}{3} \text{ for an odd number of rings, and}$$

$$M = \frac{2^{n+1} - 2}{3} \text{ for an even number of rings.}$$

Thus for a 7-ringer, $M = (2^8 - 1)/3 = 85$.

The author can remove the loop of a 7 ring puzzle, requiring 85 moves, in about 100 seconds. No doubt, with practice this can be reduced to 85 seconds,

or about a move a second. On this basis a 50-ringer will take (provided no mistakes were made) about 25 million years!

2. Electrical Chinese rings. For a long time the author wondered whether an electrical analog to this puzzle could be constructed using lamps instead of rings, with electrical relays suitably interlocked to accomplish the desired purpose. The problem, once the requirements were translated into electrical terms, turned out to be much simpler than was at first supposed. Seven small lamps with red color-caps were used, each controlled from 2 push-buttons, one to light and one to extinguish the lamp. To lend effect, a master lamp with a larger green color cap was arranged to light only when all 7 of the smaller ones were lit. Space does not permit the inclusion here of a photograph showing the device as assembled. Besides the 2 push-buttons, each lamp has 2 electrical relays associated with it, an ON relay, having 2 normally-open and one normally-closed contact and an OFF relay, which has only a single, normally-closed, contact. "Normally," in electrical parlance, means the condition when the coil of the relay carries no current.

Operation parallels the ring-puzzle conditions. A lamp lit corresponds to a ring below the loop or OFF; a lamp out corresponds to a ring above the loop or ON. The lighting of the green lamp corresponds to the loop having been completely freed. Pressing any ON button at random, will in general not light the associated lamp nor will pressing its OFF button extinguish it if it is lit. A lamp can be lit or extinguished only if the correct sequence has been followed previously and then *only by its own push-buttons.* This sequence is exactly the same as that followed in removing the loop from the rings of the Chinese puzzle.

The fundamental principle underlying the ring-puzzle solution is that, in order to take a ring off or put it on the loop, the preceding one must be on and all prior to that must be off the loop. Similarly, with the relays and lamps, in order to light any lamp the preceding one must be out (corresponding to a ring *on* the loop) and all prior to that must be lit. Thus, to light Lamp 5, Lamp 4 must be out and Lamps 1, 2 and 3, lit.

3. Electrical circuit. In these days of radio wiring diagrams, Figure 2 should readily be followed by anyone who is at all conversant with electrical circuits. All of the relays and lamps are connected similarly—their ON and OFF buttons are connected through the normally-closed contact of the preceding relay and the normally-open contacts of all relays prior to that. The connections of ON Relays 1 and 2 differ slightly from the others because they do not have a number of "prior" relays. Once a relay is energized it "seals itself in" so that it stays energized (and the corresponding lamp lit) independently of the push-button. It cannot again be extinguished unless the same circuit is set up as for lighting it. The imposing of conditions under which a lamp may be *extinguished* as well as lit by the pushing of a button requires that the button *close* a circuit rather than

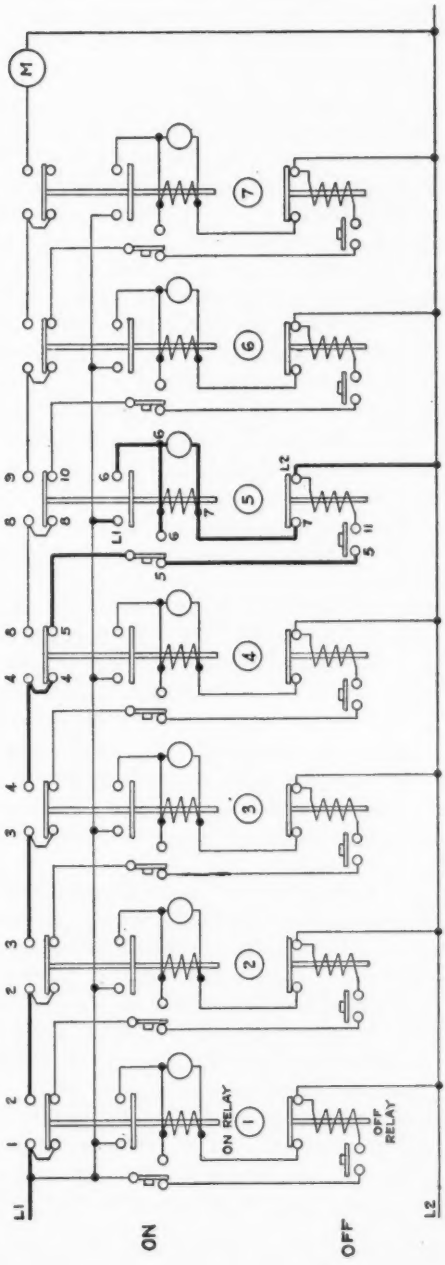

FIG. 2

open it, and since this is to *extinguish* a lamp, it is necessary to interpose an OFF relay, so that when it is *energized* by the OFF push-button, it *opens* the circuit to the lamp and ON relay.

The push-buttons associated with each relay set are shown in the diagram immediately to the left of the relays. Let us trace the circuit necessary to energize say, Relay 5 and its lamp. Starting from the incoming supply line $L2$, we trace through the "normally-closed" contact of the OFF Relay 5 to Terminal 7, through the coil of the ON Relay 5 and its lamp to Terminal 6 and to the ON push-button. Here there is no further path. Were we to press this button, we could trace across the normally-closed contact of ON Relay 4, to Terminal 4, but we could not reach $L1$ to establish a complete circuit because of the gaps 4-3, 3-2, 2-1 of the ON relays 3, 2, 1 respectively. If, however, we assume these relays are already "picked up" or energized, these gaps are closed, so that pushing ON button 5 causes its relay and lamp to operate. The relay, in turn, closes the circuit from $L1$ to 6 which now keeps it energized independently of the push-button—"sealed-in," as the electrical man says. The button can, therefore, be released and the lamp will stay lit. ON Relay 5, when energized, also opens its 8-10 contact and closes its 8-9 contact.

If, at this point, the OFF button were pushed, there would be another path from $L2$, through the coil of the OFF relay to Terminal 11 and across the OFF push-button to Terminal 5 and back to $L1$ via the same route as before. OFF Relay 5 would pick up and open the circuit across its normally-closed contact $L2$–7, thus de-energizing the ON relay and extinguishing the lamp. Releasing the button will de-energize the OFF relay and it will again close its contact, but this has no further effect on the ON relay and lamp which stay off.

If now any of the ON relays 1, 2, 3 or 4 are changed from their assumed position, there can no longer be a circuit back to $L1$ from Relay 5. Hence, pushing either No. 5 ON or OFF button can no longer light or extinguish Lamp 5. Not until the proper conditions are again established—ON Relay 4 de-energized and ON Relays 1, 2 and 3 picked up—can No. 5 relay and lamp be controlled by their push-buttons.

In tracing this circuit we assumed the proper conditions to light Lamp 5. As a matter of fact, the table shows that 5 preliminary moves would be required before Lamp 5 could be lit. Lamp 7 requires 21 preliminary moves.

This device can be built at fairly moderate cost using practically standard radio parts such as pilot lamps, midget relays and panel-mounted momentary-contact push-buttons. For a social evening, its effect is sensational and many a bridge party has been disrupted by a lady who would refuse to play until she could get the green lamp to light.

To solve the puzzle correctly, that is, to light the green lamp, the 85 moves shown in the table below must be followed. The reverse procedure is required to *extinguish* all 7 lamps—unless, of course, one were to cheat by opening the main switch.

SEQUENCE MANIPULATION OF PUSH BUTTONS TO LIGHT ALL 7 PILOT LAMPS AND THE MASTER LAMP
(85 moves required)

Move	Lamp	Move	Lamp	Move	Lamp	Move	Lamp	Move	Lamp
1.	1 on	18.	3 on	35.	1 off	52.	2 on	69.	1 on
2.	3 on	19.	1 off	36.	2 on	53.	1 on	70.	5 on*
3.	1 off	20.	2 on	37.	1 on	54.	6 on*	71.	1 off
4.	2 on	21.	1 on	38.	5 off	55.	1 off	72.	2 off
5.	1 on	22.	7 on*	39.	1 off	56.	2 off	73.	1 on
6.	5 on	23.	1 off	40.	2 off	57.	1 on	74.	3 off
7.	1 off	24.	2 off	41.	1 on	58.	3 off	75.	1 off
8.	2 off	25.	1 on	42.	3 off	59.	1 off	76.	2 on
9.	1 on	26.	3 off	43.	1 off	60.	2 on	77.	1 on
10.	3 off	27.	1 off	44.	2 on	61.	1 on	78.	4 on*
11.	1 off	28.	2 on	45.	1 on	62.	4 off	79.	1 off
12.	2 on	29.	1 on	46.	4 on	63.	1 off	80.	2 off
13.	1 on	30.	4 off	47.	1 off	64.	2 off	81.	1 on
14.	4 on	31.	1 off	48.	2 off	65.	1 on	82.	3 on*
15.	1 off	32.	2 off	49.	1 on	66.	3 on	83.	1 off
16.	2 off	33.	1 on	50.	3 on	67.	1 off	84.	2 on*
17.	1 on	34.	3 on	51.	1 off	68.	2 on	85.	1 on*

* Moves marked by an asterisk indicate final ON position of the lamps.

SKEW-SYMMETRIC MATRICES AND PROJECTIVE GEOMETRY

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General identities between two n-rowed skew-symmetric matrices P, Q with elements in a field F have been noted casually for $n=3$ and $n=4$. In case one of the matrices is regular, and therefore $n=2m$, an identity will be established here for any $m=2, 3, \cdots$. According to the application of the matrices involved, these identities can be interpreted in several different ways. As systematic use of matrix calculus seems to have found little entrance into projective geometry, the application of these identities in a sketch of the projective theory of null systems seemed to be most suggestive.

1. Matrices of rank 2. Columns with three coördinates briefly called 3-columns, representing points or vectors in the 3-dimensional space over the ground field F may be denoted by small Gothic letters:

$$\mathfrak{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \qquad \mathfrak{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

The corresponding capitals denote the associated skew-symmetric matrices

$$\mathfrak{P} = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}, \qquad \mathfrak{Q} = \cdots .$$

Then the following identity can easily be proved (*cf.* [8], p. 12):

$$\mathfrak{Q}'\mathfrak{P}\mathfrak{Q} = \mathfrak{p}'\mathfrak{q}\cdot\mathfrak{Q}, \qquad (\mathfrak{p}'\mathfrak{q} = p_1q_1 + p_2q_2 + p_3q_3), \tag{1}$$

where $\mathfrak{Q}'$ is the transpose of $\mathfrak{Q}$ and $\mathfrak{p}'$ the row having the same coördinates as the column $\mathfrak{p}$. With regard to the fact that $\mathfrak{P}\mathfrak{q} = -\mathfrak{Q}\mathfrak{p}$ is simply the so-called (outer) vector product of the two vectors represented by the columns $\mathfrak{p}$, $\mathfrak{q}$, the formula (1) is seen to be an expression for a certain threefold vector product by a simple one.

The identity (1) can be extended immediately to n-rowed skew-symmetric matrices of rank 2. Any such matrix can be written as an alternating composition of two n-columns,

$$a = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix},$$

viz,

$$[a, b] = ba' - ab'.^* \tag{2}$$

With a second matrix $[u, v]$ of this kind one obtains immediately from the definition (2) the identity

$$[a, b]'[u, v][a, b] = \delta(a, b; u, v)[a, b] \tag{3}$$

where the numerical factor $\delta = \delta(a, b; u, v)$ is defined by

$$\delta(a, b; u, v) = a'u\cdot b'v - a'v\cdot b'u. \tag{4}$$

By making use of the canonical representation of any n-rowed skew-symmetric matrix P of rank $2r$ $(r \leqq m)$ as a sum of r skew-symmetric matrices of rank 2, say

$$P = \sum_{\rho=1}^{r} [u^{(\rho)}, v^{(\rho)}] \tag{5}$$

with $2r$ linearly independent n-columns $u^{(1)}, v^{(1)}, \cdots, u^{(r)}, v^{(r)}$, one obtains from

* In the usual matrix notations ba' is the expression for the square n-matrix (of rank one if $a \neq 0$ and $b \neq 0$) with the elements $b_\mu a_\nu$ $(\mu, \nu = 1, 2, \cdots, n)$. The matrix $[a, b]$, a representative of Grassmann's "combinatorial product," is of rank 2 if and only if a and b are linearly independent; otherwise it is O. This product as well as the outer products of higher order have been introduced by Grassmann in his Ausdehnungslehre as abstract symbols with certain properties fixed by axioms. Their matricial interpretation is carried through systematically in the theory of compound matrices (*cf.* [11], Chap. V). The abstract theory in modern form, with many geometrical applications, is given in the recent book of Forder [4], Chap. VII.

(3) the following more general identity which is of the same type:

$$[a, b]'P[a, b] = \left(\sum_{\rho=1}^{r} \delta_\rho\right) \cdot [a, b] \tag{6}$$

where

$$\delta_\rho = \delta(a, b; u^{(\rho)}, v^{(\rho)}) \qquad (cf.\ (4)). \tag{7}$$

These simple results may lead to the conjecture that a non-trivial identity exists between any two skew-symmetric n-matrices; it seems, however, to be difficult to obtain further results by proceeding on this purely formal line. However, the following general relation which turns out to be useful later on, may be noted here: For any n-rowed square matrix C one has

$$[Ca, Cb] = C[a, b]C'. \tag{8}$$

2. Four-rowed matrices. To find out what type of identity could be expected in general, a preliminary investigation of the case $n=4$ seemed to be useful. In view of the decomposition

$$P = \begin{pmatrix} 0 & -p_3^{(1)} & p_2^{(1)} & p_1^{(2)} \\ p_3^{(1)} & 0 & -p_1^{(1)} & -p_2^{(2)} \\ -p_2^{(1)} & p_1^{(1)} & 0 & -p_3^{(2)} \\ p_1^{(2)} & p_2^{(2)} & p_3^{(2)} & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{P}^{(1)} & -\mathfrak{p}^{(2)} \\ \mathfrak{p}^{(2)\prime} & 0 \end{pmatrix} = \langle \mathfrak{p}^{(1)}, \mathfrak{p}^{(2)} \rangle, \tag{9}$$

the reckoning with general skew-symmetric 4-matrices is reduced to the reckoning with 3-matrices and 3-columns. Here the distinction of the last column and row is a purely technical devices which naturally disappears in all final formulas.

With Minkowski [7]* we introduce the "dual matrix"

$$P^* = \langle \mathfrak{p}^{(2)}, \mathfrak{p}^{(1)} \rangle' \tag{10}$$

of P. Then $(P^*)^* = P$ and

$$P^*P = PP^* = k(P) \cdot E \qquad (E = \text{unit matrix}) \tag{11}$$

where

$$k(P) = \mathfrak{p}^{(1)\prime}\mathfrak{p}^{(2)} \tag{12}$$

is the invariant (Pfaffian) parameter of P (and of P^*) for which

$$k(P^2) = |P|. \tag{13}$$

Thus for a regular P ($|P| \neq 0$) one has $P^* = k(P) \cdot P^{-1}$.

* In this famous work as well as in the subsequent papers of Sommerfeld [9] and Weatherburn [10] the notations are adapted to the originally intended applications of the formulae to Relativity. We leave aside this point of view here. Thus while Minkowski's calculus is invariant (or covariant) with respect to the Lorentz group, the one discussed here is orthogonal invariant (or covariant) whereby a better symmetry of the formulae is obtained. The translation of one system into the other offers no difficulty.

Let $Q=\langle \mathfrak{q}^{(1)}, \mathfrak{q}^{(2)}\rangle$ be a second skew-symmetric 4-matrix. Applying (11) to the skew-symmetric matrix $P+Q$, one can verify the identity

$$P^*Q + Q^*P = k(P, Q)\cdot E \tag{14}$$

where

$$k(P, Q) = \mathfrak{p}^{(1)\prime}\mathfrak{q}^{(2)} + \mathfrak{p}^{(2)\prime}\mathfrak{q}^{(1)} = \tfrac{1}{2}\operatorname{tr}(P^*Q) \tag{15}$$

is the common invariant ("polarized parameter") of P and Q.* In this form the identity was first established by C. E. Weatherburn ([10], p. 176, formula (28)). It can also be written in the form

$$Q'P^*Q = k(Q)\cdot P - k(P, Q)\cdot Q \tag{16}$$

which is more suitable for the purpose in hand. For $n=4$ and a singular Q we recognize in (16) the identity (6) ($k(Q)=0$).

The identity (16) involves no restrictions as to the rank of the matrices concerned. If P (and therefore P^*) is singular, say $P^*=[u, v]$, one has

$$k(P, Q) = u'Qv. \tag{17}$$

From (16) further identities can be derived, involving more than two skew-symmetric matrices. Thus we have for three, *viz.*, P, Q, R:

$$\begin{aligned} R'(Q'P^*Q)^*R = {} & (k(P, Q)\cdot k(Q, R) - k(Q)\cdot k(P, R))R \\ & - k(P, Q)\cdot k(R)\cdot Q + k(Q)\cdot k(R)P. \end{aligned} \tag{18}$$

Generally these identities show that a certain product of skew-symmetric matrices and their duals equals a linear combination of these matrices with numerical coefficients.

A dual of a skew-symmetric n-matrix cannot be defined as an n-matrix if $n>4$. Hence for the purpose of generalization of (16) the dual has to be eliminated. In fact, for a regular P this identity is equivalent to

$$k(P)\cdot QP^{-1}Q - k(P, Q)\cdot Q + k(Q)\cdot P = 0. \tag{19}$$

This relation shows the way for further generalization.

3. The general regular case. Now let P, Q be two skew-symmetric n-matrices P regular: $|P| = k_0^2 \neq 0$, $n=2m$. We consider the matrix

$$A = P^{-1}Q. \tag{20}$$

Every congruence transformation of P and Q (cogrediently) induces a similarity transformation of A, and conversely. Hence the similarity invariants of A are simultaneous congruence invariants of P and Q, and conversely.

We consider the skew-symmetric linear matrix polynomial $\lambda P-Q$ with indeterminate λ. Let $\kappa(\lambda)$ denote its invariant parameter, *i.e.*, a polynomial in λ the square of which equals the determinant

* By tr A we denote the trace (sum of the elements in the principal diagonal) of the matrix A.

$$|\lambda P - Q| = |P|\,|\lambda E - A| = k_0^2 d_A(\lambda) \tag{21}$$

where $d_A(\lambda)$ is the characteristic polynomial of the matrix A. If

$$\kappa(\lambda) = k_0\lambda^m - k_1\lambda^{m-1} + \cdots - (-1)^m k_{m-1}\lambda + (-1)^m k_m \tag{22}$$

then $k_0 = k(P)$, $k_m = k(Q)$ are the invariant parameters of P and Q, and $k_1, \cdots, k_{m-1}$ are the rational simultaneous invariants of P and Q. These numbers can be calculated explicitly by means of the similarity invariants of A because of the relation $\kappa(\lambda)^2 = k_0^2 d_A(\lambda)$.

From Cayley's well-known identity if follows immediately that $\kappa(A)^2 = 0$. This is a polynomial relation of degree $2n$ in P^{-1} and Q which in the present connection may be considered as trivial. We shall show, however, that any two skew-symmetric matrices P, Q ($|P| \neq 0$) satisfy the relation

$$\kappa(A) = 0, \tag{23}$$

or, by (22),

$$k_0 Q(P^{-1}Q)^{m-1} - k_1 Q(P^{-1}Q)^{m-2} + \cdots - (-1)^m k_{m-1} Q + (-1)^m k_m P = 0 \tag{24}$$

which, evidently, for $m = 2$ gives the identity (19).

The identity (23) follows immediately from some known theorems concerning the congruence reduction and the invariant factors of the linear matrix polynomial $\lambda P - Q$ (*cf.* MacDuffee [6], Theorems 32.2 and 32.3) whereby every invariant factor occurs at least twice as a factor of the polynomial $|\lambda P - Q|$, and thus of $d_A(\lambda)$; hence at least once as a factor of $\kappa(\lambda)$. We need this fact only for the highest invariant factor which, apart from a numerical coefficient, is the minimum polynomial $h(\lambda)$ of the matrix A. (Therefore $h(A) = 0$ from which (23) follows.*

The singular case ($k_0 = 0$) which is unimportant for the geometrical applications to be given below, requires some additional discussion of the linear matrix $\lambda P - Q$; this will be carried through in a subsequent paper.

4. Remark on vector algebra. The identity (16) proves to be the essence of what could be called "Minkowski-Grassmann calculus," *i.e.*, the natural formulation of the 4-dimensional orthogonal-invariant vector algebra (*cf.* [9] and [10]). Thus the identity (24) can be expected to yield a suitable basis for an extension

* The proof can also be based upon some statements contained in the paper of Bennet [1] where, however, irrational invariants (elementary divisors) are introduced; this would entail an unnecessary restriction of the field of the matrix elements.

Added on June 4, 1943: In a letter dated April 8, 1943 which I received on June 3, 1943, N. Jacobson informs me that after having read the abstract of a part of the present paper in the Bulletin of the American Mathematical Society, April, 1943, he noticed that the identity (23) also follows easily from a theorem obtained by him in his paper: "An application of E. H. Moore's determinant of a Hermitian matrix," Bulletin of the American Mathematical Society 45, 1939, 745–748. He mentions further that another more direct proof of his theorem has been pointed out to him by J. Williamson and that a sketch of this proof is published in the paper of N. Jacobson, "Classes of restricted Lie Algebras of characteristic p. I," American Journal of Mathematics 63, 1941, 481–515. in particular pp. 496–497. Both proofs are different from the one indicated above.

of this sort of algebra to any number n of dimensions. This has been attempted occasionally. While, however, for $n=4$ all possible cases, regular as well as singular ones, could be dealt with comprehendingly in the one relation (16) this will no longer be possible for $n>4$ when the several different singular cases will have to be considered separately. This reveals at least one of the intricacies which, for $n>4$, cause an invariant vector calculus to be unsuitable for practical application. In addition it must be noted that a complete system of relations (identities) can be stated only in terms of compound matrices.

5. The $(n-1)$-dimensional projective space. All numbers (matrix elements) occurring in the following sections will be supposed to be real. By $\mathbf{P}_{n-1}$ we denote the projective space of $n-1$ dimensions over the real field ($n>2$). A point in $\mathbf{P}_{n-1}$ is defined by a real n-column $x\neq 0$. Two points x, y are said to be equal if, and only if, the columns x, y are linear dependent or "*projectively equal*" which we indicate by writing $y\simeq x$. Thus all λx ($\lambda\neq 0$) define the same point in $\mathbf{P}_{n-1}$. Similarly two matrices A, B are projectively equal: $B\simeq A$, if, and only if $B=\lambda A$.

Linear submanifolds of $\mathbf{P}_{n-1}$ are defined in the well-known way by means of systems of linear homogeneous equations, *viz.*,

$$U_r z = 0 \tag{25}$$

where U_r is a (not necessarily square) matrix of rank r with n columns ($r<n$); or by the solution

$$z \simeq V_{n-r} t \tag{26}$$

of (25) where V_{n-r} is an n-rowed matrix of rank $n-r$ and t an arbitrary l-column if V_{n-r} has l columns (parameter-representation). If $r=n-1$ the submanifold is a point; if $r=n-2$, a straight line; here there are two linearly independent solutions of (25), say x, y. Hence the general solution can be written either in the form $z\simeq\lambda x+\mu y$ or in the form

$$z \simeq [x, y]t \tag{27}$$

where t is an arbitrary n-column [see (2)]. The matrix $[x, y]$ will be called the "*first Plücker matrix*" of the straight line. From the definition (2) it can easily be shown that it is (projectively) independent of the choice of x, y on the line.

For $r=1$ the equation (25) is equivalent to a single scalar equation

$$u'z = 0 \tag{28}$$

where u is a certain n-column $\neq 0$. This equation represents, in point coordinates z, a certain $(n-2)$-dimensional hyperplane, shortly called "the plane u." If $r=2$ the equation (25) can be replaced by two equations

$$u'z = 0, \qquad v'z = 0 \tag{29}$$

with linearly independent u, v. They represent an $(n-3)$-dimensional linear manifold in $\mathbf{P}_{n-1}$, which can be characterized by its "*second Plücker matrix*" $[u, v]$. In fact $[u, v]t$ represents the most general plane through this manifold.

If $n=4$ the latter manifold is again a straight line. What relation exists between the first and the second Plücker matrices of this line? Every pair of planes u, v having as intersection the line with the first Plücker matrix $[x, y]$, must contain the points x, y and therefore, by (29), satisfy the equations $x'u=0$, $y'u=0$, $x'v=0$, $y'v=0$ which are solved by

$$u = [x, y]^*s, \qquad v = [x, y]^*t$$

where s, t are arbitrary 4-columns. This follows from the relation

$$a[b, c]^*d = |\, a\ b\ c\ d\,| \tag{30}$$

valid for any four 4-columns a, b, c, d, since the symbol $a[b, c]^*d$ is linear and homogeneous with respect to each of the four columns, and for the unit-columns $e^{(r)}$ one has $e^{(1)}[e^{(2)}, e^{(3)}]^*e^{(4)}=1$, and so it has the characteristic properties of the determinant.

By making use of the general relation (8) we obtain

$$[u, v] = -\,[x, y]^*[s, t][x, y]^*.$$

Hence we conclude from the identity (3) that

$$[u, v] \simeq [x, y]^*. \tag{31}$$

Thus the two Plücker matrices of a given line are mutually dual. (This simple expression of an elementary fact (Jessop, [5] §4, p. 18) is mentioned here only as a typical example of the application of the identity.)

6. Collineations and correlations. A one-valued invertible point transformation $y\simeq\phi(x)$ of the space $\mathbf{P}_{n-1}$ into itself which induces a one-one correspondence $v\simeq\psi(u)$ between the planes in $\mathbf{P}_{n-1}$ such that if x runs through the plane u then y runs through the plane v, is called a *projective transformation* or *collineation*. According to the fundamental theorem of projective geometry *every collineation can be represented by a linear homogeneous transformation*

$$y \simeq Ax \tag{32}$$

where A is a certain regular n-matrix. For a proof we may refer to the paper of R. Brauer [2].* The induced plane transformation of (32) is readily seen to be

$$v \simeq A'^{-1}u. \tag{33}$$

A one-one correspondence between points and planes in $\mathbf{P}_{n-1}$ which with every point x associates a plane $v\simeq\chi(x)$ and with every plane u a plane $y\simeq\omega(u)$ is said to be a *correlation* if, when x runs over all points of a plane u, the plane $v\simeq\chi(x)$ runs through all planes passing through the point $y\simeq\omega(u)$. By carrying out two correlations in succession one obtains a collineation. This means that if $u^{(1)}\simeq\chi^{(1)}(y)$, $x^{(1)}\simeq\omega^{(1)}(v)$ is a second correlation, then

* Here a somewhat more general theorem is proved which, under our present suppositions, gives exactly the above statement because the real field admits no automorphism different from the identity. For another proof see for instance [3], p. 27–32.

$$x^{(1)} \simeq \omega^{(1)}(\chi(x)), \qquad u^{(1)} \simeq \chi^{(1)}(\omega(u))$$

is a collineation. If as the second correlation we take in particular $u^{(1)} \simeq y$, $x^{(1)} \simeq v$ we see that $x^{(1)} \simeq \chi(x)$, $u^{(1)} \simeq \omega(u)$ is a collineation .Thus the fundamental theorem shows that *for every correlation* $v \simeq \chi(x)$, $y \simeq \omega(u)$ *a regular n-matrix C can be found such that it appears in the form*

$$v \simeq Cx, \qquad y \simeq C'^{-1}u. \tag{34}$$

Any correlation is seen to be invertible; the inverse of (34) is

$$u \simeq C'y, \qquad x \simeq C^{-1}v. \tag{35}$$

Now let B be the matrix of another correlation; the product of both is the collineation with the matrix

$$A \simeq B'^{-1}C. \tag{36}$$

Therefore two correlations are said to be commutative if

$$C'^{-1}B \simeq B'^{-1}C. \tag{37}$$

The square of the correlation (34) is the collineation with the matrix $C'^{-1}C$. The correlation C (*i.e.* (34) will be said to be *involutory* (or "of period 2") if its square is the identity, *i.e.*, if $C'^{-1}C \simeq E$. This means $C' \simeq C$ or $C' = \lambda_0 C$. By taking the determinants of both members of this equation one sees that $\lambda_0 = +1$ if n is odd, $\lambda_0 = \pm 1$ if n is even. Hence we see that *the matrix C of an involutory correlation in* $\mathbf{P}_{n-1}$ *is always symmetric if n is odd; it is symmetric or skew-symmetric if n is even.* A correlation with skew-symmetric matrix (which is naturally always involutory) is called a *null system.** Evidently null systems can exist only in odd-dimensional spaces $\mathbf{P}_{n-1}$.

Often it is useful to present a correlation in another algebraic appearance. In "running coördinates" y the equation of the plane v (*i.e.*, $v'y = 0$) that is associated with the point x by the correlation (34), is

$$x'C'y = 0. \tag{38}$$

In running plane coördinates v the equation of the point associated to the plane u is

$$u'C^{-1}v = 0. \tag{39}$$

This form of the definition of a correlation is particularly convenient for the study of its behaviour by means of projective coördinate transformations or correlations. When new coördinates are introduced by the substitution with the matrix S, so that $x \simeq S\bar{x}$, $y \simeq S\bar{y}$ the equation (38) is changed into $\bar{x}'S'C'S\bar{y} = 0$ whence we see that in the new coördinates the correlation has the matrix

$$\overline{C} \simeq S'CS. \tag{40}$$

* R. Brauer [2] has given another, more geometric, characterization of null systems.

Similarly if we apply in $\mathbf{P}_{n-1}$ the correlation

$$x \simeq R'^{-1}\bar{u}, \qquad y \simeq R'^{-1}\bar{v}$$

the equation (38) is turned into $\bar{u}'R^{-1}C'R'^{-1}\bar{v}=0$ which defines a correlation in plane coördinates. With respect to (39) we see that, in point coördinates, this correlation would have the equation $\bar{x}'\overline{C}'\bar{y}=0$, its matrix being

$$\overline{C} \simeq (R^{-1}CR'^{-1})'^{-1} = R'C'^{-1}R. \tag{41}$$

By comparison with (37) we obtain the following theorem:

A correlation C is invariant under the correlation R of the coördinates (i.e. $\overline{C}\simeq C$ in (41)) if, and only if, these two correlations are commutative.

7. Null systems. After these preliminaries we now turn to a more detailed consideration of the correlations with skew-symmetric regular matrix P. Accordingly we suppose $\mathbf{P}_{n-1}$ to be of odd dimension $\geqq 3$, hence $n=2m$. In view of (38) and (39) the null system P is defined by the equation

$$x'Py = 0 \tag{42}$$

or by its dual

$$u'P^{-1}v = 0. \tag{43}$$

It constitutes a dual relation between point z and plane $w\simeq Pz$. Both are always coincident since, because of the skew-symmetry of P, one has for all x

$$x'Px = 0. \tag{44}$$

This property is characteristic of a null system. In fact, if the correlation with the matrix C satisfies the condition $x'Cx=0$ for all x, then C is necessarily skew-symmetric. To prove this, let $C=A+P$ where A is symmetric, P skew-symmetric. By (44) one has $x'Ax=0$ for all x, whence $A=0$ and thus $C=P$.

Now let z move through the $(n-3)$-dimensional manifold defined by its second Plücker matrix $[u, v]$. The corresponding two-dimensional plane $w\simeq Pz$ then runs over all planes containing the straight line through the two points $\bar{x}\simeq P^{-1}u$, $\bar{y}\simeq P^{-1}v$; hence the first Plücker matrix of this line is by (8)

$$[\bar{x}, \bar{y}] \simeq P^{-1}[u, v]P^{-1}. \tag{45}$$

The result of the dual consideration is contained in this formula.

If $n=4$ the manifold $[u, v]$ is likewise a straight line. Here the identity (16) leads to a very simple transformation formula for the Plücker matrices of the straight lines related by the duality of the null system. By applying the identity to the right-hand member of (45) (after having replaced P^{-1} by $P^*\simeq P^{-1}$) and making use of (17) we obtain

$$[\bar{x}, \bar{y}] \simeq u'P^*v \cdot P^* - k_0[u, v]^* \tag{46}$$

where k_0 is the invariant parameter of P. By dualization of (46) we get the second matrix of the same line

$$[\bar{u}, \bar{v}] \simeq u'P^*v \cdot P - k_0[u, v] \tag{47}$$

which is said to be *conjugate* (with respect to the null system P) to the line with the second matrix $[u, v]$. Similarly a transformation law can be established for the first Plücker matrices of the straight lines conjugate with respect to P:

$$[\bar{x}, \bar{y}] \simeq x'Py \cdot P^* - k_0[x, y]. \tag{48}$$

From these formulae we see immediately that a line $[x, y]$ (or $[u, v] \simeq [x, y]^*$) coincides with its conjugate if, and only if

$$x'Py = 0, \qquad u'P^*v = 0. \tag{49}$$

These lines are called self-conjugate or *null lines* for the null system P. From (49) we conclude that a straight line defined by two of its points (or as the intersection of two planes) is a null line for a given null system if and only if both of these points (or planes) coincide with the two planes (or points) which are associated with them by the null system.

Now let

$$P = [u^{(1)}, v^{(1)}] + [u^{(2)}, v^{(2)}] \tag{50}$$

where $u^{(1)}, v^{(1)}, u^{(2)}, v^{(2)}$ are four linearly independent 4-columns. *Then the matrices* $[u^{(1)}, v^{(1)}]$, $[u^{(2)}, v^{(2)}]$ *are necessarily the second Plücker matrices of a pair of conjugate non-null lines of the null system* P. To prove this we consider the two points $x \simeq P^*u^{(1)}$, $y \simeq P^*v^{(1)}$ on the line conjugate to the line with the second Plücker matrix $[u^{(1)}, v^{(1)}]$. By (30) we have

$$x \simeq [u^{(2)}, v^{(2)}]^*u^{(1)}, \qquad y \simeq [u^{(2)}, v^{(2)}]^*v^{(1)}$$

and therefore the first Plücker matrix of the conjugate line is

$$[x, y] \simeq [u^{(2)}, v^{(2)}]^*[u^{(1)}, v^{(1)}][u^{(2)}, v^{(2)}]^* \simeq [u^{(2)}, v^{(2)}]^* \qquad \text{(by (8) and (3))}$$

whence by (31) we conclude that $[u^{(2)}, v^{(2)}]$ is its second Plücker matrix.

From (47) it follows that a null system P is completely defined by its invariant k_0 and a pair of conjugate non-null lines.

8. Null systems in involution. Two null systems P, Q in P_{n-1} are said to be "in involution" if they are

1. different, *i.e.*, the matrices P, Q are linearly independent;
2. commutative, which by (37) means

$$P^{-1}Q \simeq Q^{-1}P, \quad \textit{i.e.}, \quad P^{-1}Q = \alpha Q^{-1}P \tag{51}$$

with a certain real number α. If we introduce again the matrix A of (20), this condition means that the matrix A, being not a scalar multiple of E itself, is (projectively) involutory:

$$A^2 = \alpha E. \tag{52}$$

Hence α can be calculated by the rational similarity invariants of A; let

$$d_A(\lambda) = |\lambda E - A| = \lambda^n - a_1\lambda^{n-1} + \cdots + (-1)^n a_n = \prod_{\nu=1}^{n} (\lambda - \alpha_\nu)$$

where $\alpha_1, \cdots, \alpha_n$ are the characteristic roots of A. Then

$$a_1 = \operatorname{tr} A = \sum_{\nu=1}^{n} \alpha_\nu, \qquad a_2 = \sum_{\mu<\nu} \alpha_\mu\alpha_\nu$$

and

$$\alpha = \frac{1}{n} \operatorname{tr}(A^2) = \frac{1}{n} \sum_{\nu=1}^{n} \alpha_\nu^2 = \frac{1}{n}(a_1^2 - 2a_2).$$

Therefore α can be expressed by means of the rational simultaneous invariants k_0, k_1, k_2 of P and Q (in the notations of (22)). Because of

$$a_1 = 2\frac{k_1}{k_0}, \qquad a_2 = \frac{1}{k_0^2}(k_1^2 + 2k_0k_2)$$

one has

$$\alpha = \frac{1}{mk_0^2}(k_1^2 - 2k_0k_2). \tag{53}$$

Suppose P, Q to be two null systems in involution. It follows from (52) that

$$(P^{-1}Q)^{2i} = \alpha^i E, \qquad (P^{-1}Q)^{2i+1} = \alpha^i P^{-1}Q = \alpha^{i+1}Q^{-1}P.$$

By putting these expressions into the general identity (24), all matrix products herein are reduced to scalar multiples of P or Q. We get a relation of the form

$$f(\alpha)Q - g(\alpha)P = 0 \tag{54}$$

where $f(\zeta)$ and $g(\zeta)$ are two polynomials in ζ: For $m=2l+1$ (l any positive integer)

$$f(\zeta) = k_0\zeta^l + k_2\zeta^{l-1} + \cdots + k_{2l},$$
$$g(\zeta) = k_1\zeta^l + k_3\zeta^{l-1} + \cdots + k_{2l+1};$$

for $m=2l$

$$f(\zeta) = k_1\zeta^{l-1} + \cdots + k_{2l-1},$$
$$g(\zeta) = k_0\zeta^l + k_2\zeta^{l-1} + \cdots + k_{2l}.$$

From (54) and the linear independent of P and Q we infer that *the number α given by* (53) *must be a root of both of these polynomials $f(\zeta)$, $g(\zeta)$*:

$$f(\alpha) = 0, \qquad g(\alpha) = 0. \tag{55}$$

These equations are necessary conditions for two null systems to be in involution.*

* The apparent difference between the conditions for even and for odd values of m may only be noted here. It emerges again in the theory of the symplectic groups associated with the null systems which will be dealt with in another paper.

For $n=4$ $(m=2)$ the polynomial $f(\zeta)$ is the constant k_1. We have the condition

$$k_1 = k(P, Q) = 0 \tag{56}$$

which proves to be the only one since $\alpha = -k_2/k_0$ satisfies the equation $g(\alpha)=0$. From (19) it follows that the condition (56) is likewise sufficient. Supposing P in the form (50) and correspondingly $Q=[a^{(1)}, b^{(1)}]+[a^{(2)}, b^{(2)}]$, one can write the simultaneous invariant of P and Q in the form

$$k_1 = \left| u^{(1)}v^{(1)}a^{(1)}b^{(1)} \right| + \left| u^{(1)}v^{(1)}a^{(2)}b^{(2)} \right| + \left| u^{(2)}v^{(2)}a^{(1)}b^{(1)} + \left| u^{(2)}v^{(2)}a^{(2)}b^{(2)} \right| \right.$$

whence because of (30) follows the usual geometric characterization of two $\mathbf{P}_3$-null-systems in involution: Two null systems P, Q in $\mathbf{P}_3$ are in involution if and only if the conjugate with respect to P of every non-null line of P which is a null line of Q is also a null line of Q.

For $n=6$ $(m=3)$ the two conditions (55) are

$$f(\alpha) = k_0\alpha + k_2 = 0, \qquad g(\alpha) = k_2\alpha + k_3 = 0$$

whence by means of (53)

$$k_2 = -\frac{k_1^2}{k_0}, \qquad k_3 = -\frac{k_1^3}{k_0^2}, \qquad \alpha = \frac{k_1^2}{k_0^2}$$

and

$$k_{0\kappa}^2(\lambda) = (k_0\lambda - k_1)^2(k_0\lambda + k_1).$$

Thus here the characteristic roots of $A=P^{-1}Q$ are $\pm k_1/k_0$ with multiplicities 4 and 2 respectively.

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MATHEMATICS, 200 B.C.–600 A.D.

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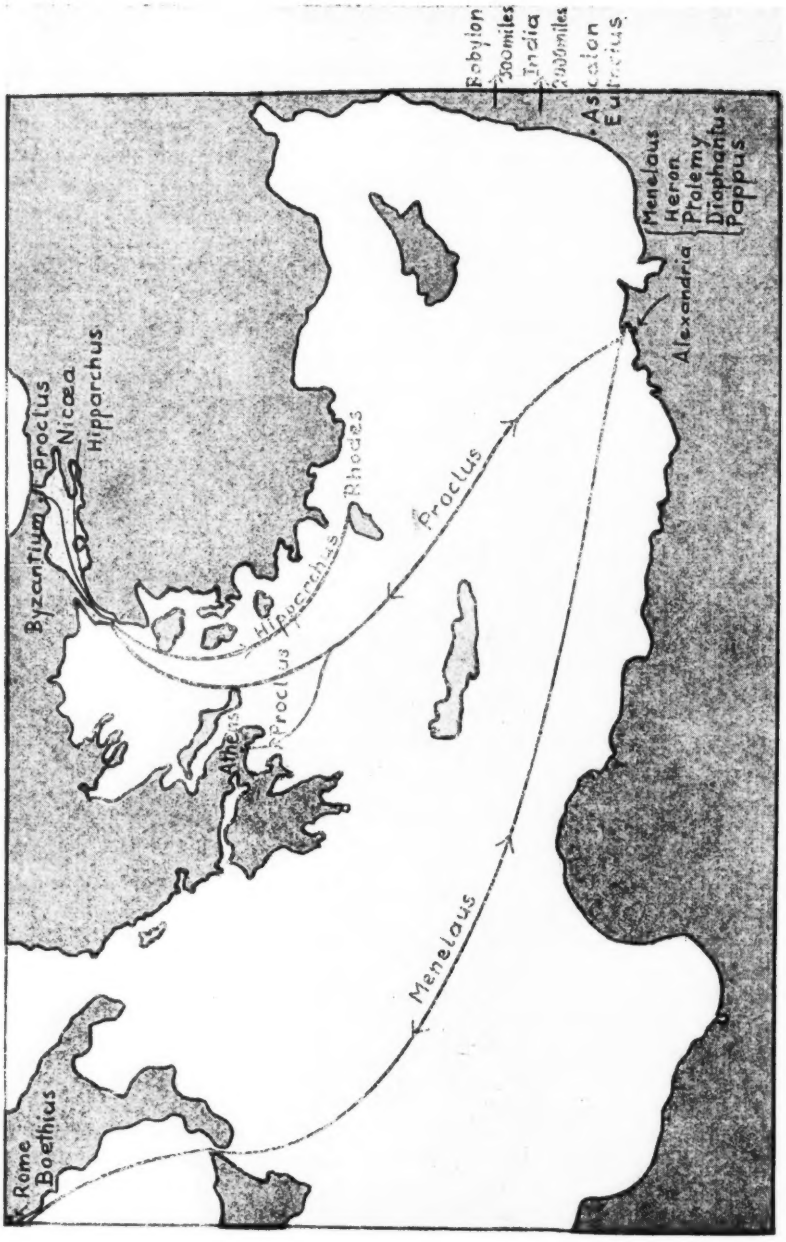
1. Trigonometry. It is in this period that we find Greek mathematics strongly influenced for the first time by phenomena outside the world of mathematical ideal entities. The astronomer observes the positions of the sun, moon, planets and fixed stars at different places and at different times. He measures a number of angles which are not independent variables. To establish their relations, to determine other, not observable, angles, for instance the angle measuring the arc between the annual circular path of the sun and the pole of the daily circles of the fixed stars, requires new mathematical investigations. All these problems pertain to the geometry of the sphere and the solution of these problems constitute what is now called spherical trigonometry.

Furthermore, we find plane trigonometry developed. The investigation of relations between the sides and the angles of a plane triangle was perhaps inspired by spherical trigonometry.

Now, in the case of spherical trigonometry, as in that of plane trigonometry, the relations are not algebraic if one measures the angles as fractional parts of a full angle, which is always the case in astronomical observations. One needs to introduce non-algebraic functions of the fractions determining the angles in order to obtain algebraic relations. It is sufficient to introduce one function of this kind: One may take the angle as an angle between the radii of a circle, then this function can be chosen as the ratio of the chord subtending the angle to the diameter of the circle.

Thus we have two problems for spherical and plane trigonometry: (1) One has to find the (algebraic) relations between the chord function of the angles and the arcs in a spherical triangle and the (algebraic) relations between the chord function of the angles and the sides of a plane triangle. (2) One has to investigate the chord function, which again may be divided into three parts: (a) the determination of the algebraic functional relations; (b) the numerical computations of the function, which in turn implies (c) the finding of certain inequalities determining the behavior of the function in the neighborhood of certain points. We note in passing that the chord function of an angle is double the sine function of the half of the angle.

2. The seeds of the notion of function and of transformation. In trigonometry the mathematicians came closer to the notion of function than in the older theory of "locus." But the general idea of function did not appear at all, still less the idea of transformation. A transformation is, one may say, nothing else than a materialized function: the transformed geometric element, for instance a new point, is a function of the old one. But this embodiment of a function is not so easily visualized as the locus. We shall see below that the main difficulties confronting the foundation of projective geometry are overcome in this period. But still we do not find the beginning of a systematic development of projective


Babylon
300miles
India
Menelaus
Heron
Ptolemy
Diophantus
Pappus
Alexandria
Byzantium
Proclus
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Rhodes
Hipparchus
Proclus
Proclus
Athens
Menelaus
Rome
Boethius

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geometry. We do not even find a systematic use of the fact that the conics are generated by the projective transformation of a circle.

3. Commentaries. In this period we find several valuable commentaries on the works of the mathematicians Euclid, Archimedes, and Apollonius.

4. Other peoples. The disturbances of the old world by the expeditions of Alexander had also an effect on the state of mathematics. The Greeks came into closer contact with the Egyptians and with the people of Mesopotamia. The inhabitants of India came in contact with Greek art and Greek mathematics, also in closer contact with the culture of Asia Minor. Although Greek mathematics during its first period was certainly under Babylonian influence, its peculiar and vigorous development obscured that influence for many centuries. At this time, however, the characteristics of Babylonian mathematics became quite apparent. Mathematical exercises, to be seen in cuneiform texts from 2000 B.C., appear in Greek textbooks. In India, we see a flourishing of mathematics under Greek influence, especially arithmetic which, perhaps, was reflected back to the Greeks. For the first time after Euclid we find in this period new arithmetical problems and new methods for their solution. These were again seeds which, 1500 years later, were developed into full growth.

Important new symbols for numbers were introduced in India and were transmitted to the Near East and Europe where we shall encounter them in the next period.

5. General significance of this period. The time covered by this article is very long, 800 years, in comparison with the periods covered by the former reports. In it no mathematician of such glorious fame as Euclid, Archimedes, or Apollonius appears. But it is not easy to call it a period of decadence if one recognizes the many seeds to be developed later. One mathematician of this time, Pappus (about 300 A.D.) even expressed his feeling that mathematics up to his time was only in the beginning of its development. He says: "I saw that all ⟨mathematicians⟩ move about only in the beginnings of pure and applied mathematics; and I had a feeling of awe ⟨because I was aware⟩ that I could show much better and much more useful things." Perhaps Pappus was afraid to enter alone into the vast and unknown realm of that science the existence of which he divined. Already in Pappus' time the best minds were more interested in mystical or theological problems than in scientific ones.

The great treatise of Ptolemy of Alexandria (about 150 A.D.) on astronomy stood for more than 1500 years by the side of Euclid's Elements as a book of indisputable authority.

6. The foundations of trigonometry. This treatise of Ptolemy is commonly called *Almagest* into which word the Arabs changed the original title, *ἡ μεγάλη σύνταξις*, The Great Composition. In this book Ptolemy collected, enlarged and systematized the results of preceding astronomical investigations, both practical and theoretical, especially those of the great Greek astronomer Hipparchus

(about 200 B.C.). Most famous is his systematization of the movement of the planets by using combinations of circular movements. But this part of his work is of minor interest for the history of mathematics.

The trigonometry of the *Almagest* is based on two theorems. The first is called the theorem of Menelaus, who lived about fifty years before Ptolemy in Alexandria. His work is extant only in Arabic and Hebrew translations. The figure represents the theorem of Menelaus for plane figures where we have the relation

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.$$

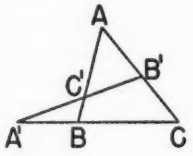


FIG. 1

This theorem is also valid for the sine function of the arcs on the sides of a spherical triangle and in this form is used by Ptolemy to prove the relations between the arcs and angles of a spherical triangle. It is interesting to see why the theorem for the plane figures is so easily changed into a theorem for spherical figures: the theorem in the plane is easy to generalize for the plane of projective geometry by taking cross ratios instead of ratios. Then we obtain immediately the corresponding theorem about planes and lines through a point which we take as center of the sphere. The cross ratio of points on a line in the plane corresponds to the cross ratio of lines in a plane through the central point. And this latter cross ratio is expressed by the sine function of the angles between the lines; hence the advantage of our sine function in comparison with the original chord function. Menelaus, of course, as well as Ptolemy, does not go the way of projective geometry to prove the spherical theorem by the plane theorem.

The second theorem is used to find the functional relations for the chord function. This theorem is called after Ptolemy and states a relation between the six distances of four points on a circle. The proof of Ptolemy is probably the shortest possible, and is to be found in all textbooks on geometry. The analysis of the theorem shows the following elements: first, an identity between two cross ratios, determined by the same four points, an identity known as Euler's identity for four points on a line; second, the algebraic identity expressing the invariance of the cross ratio under a linear transformation; third, the geometric fact that a linear transformation in the plane of a complex variable transforms lines into circles. The theorem is used in the special case where two of the four points are on one diameter. To go, by way of this theorem, to the addition theorem for the sine function is certainly not the simplest possible way.

For the numerical evaluation of the chord function Ptolemy needs the addition theorem and further the fact that the function $(\sin x)/x$ is decreasing with increasing x for $0<x<\pi/2$. The fact that $(\tan x)/x$ is increasing with increasing x for $0<x<\pi/2$ is already proved in a very simple way in Euclid's Optics. One may prove in a similar and quite as simple way that $(\sin x)/x$ is decreasing with increasing x for the same range. Ptolemy gives a rather complicated proof. The fact itself had already been used by Aristarchus of Samos more than three hundred years before Ptolemy when he discussed the appearance of the sphere of the moon. Tables for the chord function were given long before Ptolemy by Hipparchus but these have not been handed down to us.

7. Achievements in geometry. The most remarkable achievements of this period in geometry are due to Pappus. In the seventh book of his "Mathematical Collections" he proved the theorem that the cross ratio of four points on a line $(AC/AD):(BC/BD)$ is not altered by perspective projection. With the help of this theorem he proves several other propositions. By far the most important is the following: Let A, B, C, be three points on one line, A', B', C', three points on another line, then the lines AB' and BA', BC' and CB', CA' and AC', respectively, meet in three points lying on one line.

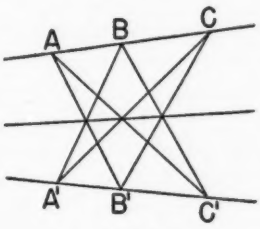


FIG. 2

This theorem marks an event in the history of geometry. From the beginning geometry was concerned with measures: lengths of lines, areas of plane figures, volumes of bodies. Here we have for the first time a theorem which is established by the ordinary theory of measures but is itself free of all elements of measurement; it states the existence of a figure which is determined through the incidence of lines and points only. It is the first "configuration" of projective geometry, and it was shown more than 1500 years later that this configuration alone is sufficient to build up projective geometry in the plane.—The mathematicians who pointed out the important role of this theorem were unjust toward Pappus in naming the theorem after Pascal.

8. Practical mathematics. After Euclid's Elements and Ptolemy's Almagest there is perhaps no ancient work on mathematical methods and natural science which had such a lasting, uninterrupted influence as that of Heron. About Heron's life we know scarcely anything. For a long time scholars tried to find out when he lived through a study of his works, and made guesses ranging from

150 B.C. to 200 A.D. For the moment it seems to them most probable that he lived in Alexandria in the first century A.D. There are books bearing his name which probably have not been written by him. In this whole collection we find little pure mathematics. It is only incidentally that, teaching all sorts of practical methods of measuring, he states and proves rigorously how to express the area of a triangle in a symmetrical way in terms of the sides. He also shows himself as a resourceful mathematician in the proof of the theorem that the three auxiliary lines in Euclid's proof of the Pythagorean theorem meet in one point. But the main tendency of Heron is to make mathematics bear fruit in the treatment of practical problems. He shows many ways of finding approximate measures.

He also treats problems of surveying, for instance the problem of finding the direction of the line joining two points, if one of the points is not visible from the other point. He solved this problem by measuring the rectangular coordinates of auxiliary points between the two given points. This is in symbols: x_0, y_0 and x_n, y_n may be the coordinates of the two given points, x_i, y_i $(i=1, \cdots, n-1)$ the coordinates of the auxiliary points; then the direction is given by

$$\frac{y_n - y_0}{x_n - x_0} = \frac{\sum_{i=1,\cdots n} (y_i - y_{i-1})}{\sum_{i=1,\cdots n} (x_i - x_{i-1})}.$$

The idea of determining the position of different points on a surface, especially the earth, by two coordinates is much older than its appearance in Heron's work. Coordinates were quite indispensable to the astronomer in determining the relative positions of the places of observation. Thus we find already Hipparchus determining the points on the globe through the two angles of latitude and longitude.

In many of the problems of Heron concerned with measuring we find similarities with Egyptian methods. There are other aspects of his work which are obviously connected with the mathematics of the Babylonians. Thus we find an example of a quadratic equation for the radius of a circle given through the sum of the number measuring the area of the circle and of the number measuring the circumference. Such problems as this have scarcely any practical value; furthermore, they are very remote from the problems of classical Greek mathematics. Similar problems, however, are to be found in very old collections of Babylonian exercises.

Heron also tries to compute the volume of a truncated pyramid whose linear measures as they are given are impossible. Out of these data he gets for the volume an expression corresponding to the square root of a negative number. He takes instead of this expression the square root of the number with positive sign, which magnitude, within the given problem, has no significance whatsoever.

We may say that there are in Heron's works characteristics of the Greek, Egyptian, and Babylonian mathematics, and even a premonition of developments of much later times.

Heron was not only an ingenious mathematician. He observed the forces of nature and used them to build all sorts of machines, most of them of very little practical use. He was probably the first to use the power of expanding steam to set heavy bodies in motion, for instance to make a sphere rotate about an axis. His manifold inventions were well known to the Renaissance physicists, among others, to Galileo.

9. Algebra and theory of numbers. There is a fourth mathematician in this period whose name still lives in the work of modern mathematicians, Diophantus. In the first book of his Arithmetic we find many problems not very different from old Babylonian problems. But the form of his treatment of these problems is very important: as the problems never lead to an irrational quantity, Diophantus is able to present a theory of equations in a seemingly modern form. All the difficulties in operations with irrational quantities, which can only be overcome by an at least partially developed theory of limits, do not appear here. Thus we find here symbols and methods of solution of equations quite similar or equivalent to modern symbols and methods. The form of Diophantus' work has undoubtedly influenced the further development of algebra. However, he was probably not the first to give this form to arithmetical operations.

We know nothing of the life of Diophantus. Those scholars may be right who suppose that Diophantus lived at Alexandria in the time of Ptolemy and Heron.

Beginning with the second book of Diophantus' Arithmetic we find a new type of problem, belonging to the theory of numbers. In the first problem of this type one has to find two rational numbers which squared and added are equal to a given square, a^2. The method of the solution is quite general. He puts one number equal to x, the other equal to $rx-a$. Then he finds $x=2ar/(r^2+1)$. He indicates the general solution but takes special values for a and r. This problem is, of course, not different from the old problem of finding two rational numbers x and y so that $x^2+y^2=1$. But already the next problem is something new: to divide the sum of two squares a^2+b^2 into two other squares. He puts one number equal to $x+a$ and the other number equal to $rx-b$ and finds $x=(2rb-2a)/(r^2+1)$. Again he indicates a general solution but takes special values for a, b, and the parameter r. This procedure is nothing else than the rationalization of the equation $u^2+v^2=a^2+b^2$.

In the same way the next example is to be considered as a rationalization of the algebraic relation $y^2-z^2=d$. In general we may characterize the problems of Diophantus as problems of rationalization of algebraic relations, *i.e.*, of the finding of a representation of an algebraic relation through rational functions of a parameter. Diophantus does not try to find solutions in integers.

These problems and their solutions made a great impression on mathematicians of the sixteenth and the seventeenth centuries and were certainly one of the reasons for a new flowering of that noble science, the theory of numbers.

10. Commentaries. The most famous of the ancient commentators is doubtless Proclus, who lived in the middle of the fifth century A.D. But he was much more a philosopher than a mathematician. He was head of the Neo-Platonic school at Athens. His contributions to mathematics are certainly very slight, but his commentary on the first book of Euclid is an invaluable source for the history of Greek mathematics. The commentary is also typical of his time, which considered metaphysical speculations, mostly of a mystical character, as the most important task for lovers of wisdom. These people looked down on mathematics because it made use of hypotheses whereas pure speculation was non-hypothetical.

Heron, too, made additions to, and commentaries on, the work of Euclid. Pappus wrote about Euclid, Apollonius and other mathematicians. Of the later commentators we mention Eutocius, who lived in the sixth century A.D. and came from Ascalon in Syria. He showed himself a very able mathematician and gave us an excellent commentary on Archimedes, where we find the most valuable report on the different solutions of problems of the third degree. He was even able to restore an old corrupted manuscript, probably of Archimedes, where we find the solution of a maximum problem, mentioned in our third report.

Here we make an end to our necessarily incomplete report on Greek mathematics.

11. The Romans. We mention only one Roman, Boethius, who was executed by Theodoric, King of the Goths, in 524. His mathematical importance lies in his role as translator. In his time Greek was no longer known by all who were interested in science. Boethius translated, along with Plato and Aristotle, also Euclid, Ptolemy, and Archimedes, but these translations are not extant. He is the first author where we find the quadruple of the four sciences, arithmetic, music, geometry, and astronomy, as constituting the mathematical branch of the liberal arts, the quadrivium. We do not owe to the old Romans any significant contribution to mathematical science.

12. Mathematics in India. We can give only a very short review of mathematical activities in India during this period. The outstanding mathematicians of this country were primarily astronomers; we mention only Aryabhatta (about 500 A.D.) and Brahmagupta (about 600 A.D.). Whereas Diophantus treated problems of finding rational values for algebraic relations of second and higher degree, the mathematicians of India treated the problem of finding integers satisfying linear relations. One recognizes that such problems are indeed of interest in astronomical investigations. It is probable that the Indians were depending on the research work of Babylonian astronomers.

Probably the Indians, in their development of a new symbolism for the writing of integers, were also indebted to the Babylonians. They determined the integer by expanding it into a power series with 10 as basic number: $n=\sum_{i=0}^{i=m} a_i 10^i$, $(a_i<10)$. Then they represented the integer by the symbol $a_m \cdots a_2 a_1 a_0$. The old Babylonians had already represented integers in this way,

using 60 as the basic number. But their representation was ambiguous because they did not use symbols for those coefficients which are equal to zero.

We find our sign 0 already in the astronomical tables of Ptolemy, and we also find it used as abbreviation for "nothing" (*οὐθέν*) in Heron's writings. But the systematic use of the symbol 0 came from India to the Arabs and through them to Europe, and had an inestimable influence upon all kinds of scientific and practical computations.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1942-43

Kappa Mu Epsilon, Mount St. Scholastic College

The Kansas Gamma Chapter held regular monthly meetings throughout the year. Among the most interesting programs presented at the meetings were the following:

Mathematics and its place in national defense, a panel discussion in which special emphasis was placed on specific mathematical requirements for various branches of the service and how collegians can best promote the war effort.

Quiz Kids, a program to which the faculty and student body were invited, and at which an intensive study of cultural, disciplinary and practical aspects of mathematics enabled the "Quiz Kids" to answer questions proposed by an interested audience.

Famous women in mathematics, a round table discussion at which important mathematicians from various European countries were discussed in chronological order.

The two major social events of the year were the annual Christmas party and the formal tea in May which followed the initiation of new members. Officers were: President, Margaret Molloy; Vice-President, Mary Margaret Downs; Secretary, Virginia Meyers; Treasurer, Mary Margaret Walters; Faculty Sponsor, Sister Helen Sullivan, O. S. B. The officers elect for 1943–44 are: President, Virginia Meyers; Vice-President, Rosemary Solas; Secretary, Rosa Garcia; Treasurer, Mary Margaret Walters; Faculty Sponsor, Sister Helen Sullivan, O. S. B.

Aphesteon, Honorary Mathematics Association of the New Jersey State Teachers College at Montclair

Due to the accelerated program of education and the fact that many members left during the year for various branches of the armed services, the meetings of *Aphesteon* have been irregular. Among the papers presented were the following.

Complex roots, by Dr. Howard Fehr

Non-Euclidean geometry, by Ruth Wheeler

A cryptographic application to linear equations, by Dr. Albert Meder, New Jersey College for Women

Linkages, by George Kays.

Another program, devoted to the subject of aeronautics, was presented with the aid of films on map projections and celestial navigation. Strip films on pilotage were also shown. The introductory and between-film comments were provided by Dorothy De Witt. *Aphesteon* also took part in making the annual spring picnic of the Mathematics Department a success. Prizes for highest scholastic records in mathematics were awarded to John Macchi and Philip Egeth who graduated in June 1942 and are now with the armed forces. Officers for the first and second terms were: Presidents, Ruth Wheeler (I) and Dorothy De Witt (II); Secretary-Treasurers, Mary Casbarro (I) and Laurel Neild (II); Adviser, Dr. Virgil Mallory.

Delta X Mathematics Club, University of Toledo

Topics considered by undergraduate speakers included:

Geometrical applications of complex numbers, by William Landry

The life of Newton and the development of the calculus, by Ralph McBee

Time, by Julian Bulley

Dimensional analysis, by Philip O'Neill

Mil measure, by Edward Faneuf

Lineal elements, by T. R. Hunter

Continued fractions and their use, by John Mason

Geometrical constructions with rule alone and with compass alone, by Ernest and John Weaver.

Each regular meeting was followed by a social hour with mathematical games for entertainment and with refreshments. The club held also two picnics, several special parties and the annual banquet, at which a talk was given on the subject:

Discontinuous functions, by Professor Fern Welker.

Officers for the year were Presidents, Charles Delbecq and then William Landry; Vice-President, Julian Bulley; Secretary-Treasurer, Frederick Racker. Professor Wayne Dancer has been faculty adviser to the Club since its founding in 1929. The membership this year was eighty-five.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SLIDE RULE SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

F. L. DENNIS, Ursinus College

The following method of solving oblique spherical triangles is well adapted to a short and fairly accurate slide rule computation.

Assuming the law of cosines in the form

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha, \tag{1}$$

substitute

$$\alpha_b = \arctan (\cot b/\cos \alpha), \qquad \left(-\frac{\pi}{2} < \alpha_b < \frac{\pi}{2}\right), \tag{2}$$

$$r = \sqrt{\cos^2 b + \sin^2 b \cos^2 \alpha} = \cos b/\sin \alpha_b. \tag{3}$$

Dividing (1) by r and substituting from (3), we obtain

$$\frac{\cos a}{\cos b} = \frac{\sin (c + \alpha_b)}{\sin \alpha_b}. \tag{4}$$

This can be written in the form

$$\frac{\sin (90 - a)}{\sin (90 - b)} = \frac{\sin (c + \alpha_b)}{\sin \alpha_b}, \tag{5}$$

where

$$\tan \alpha_b = \frac{\tan (90 - b)}{\sin (90 - \alpha)},$$

which may be referred to as the auxiliary law of sines, since it involves the auxiliary angle α_b.

There are six forms of the auxiliary law of sines obtained by permuting the sides in all possible ways. There are six corresponding forms which can be derived from the law of cosines for angles, of the type

$$\frac{\sin (90 - \alpha)}{\sin (90 - \beta)} = \frac{\sin (\gamma - a_\beta)}{\sin a_\beta}, \tag{6}$$

where

$$\tan a_\beta = \frac{\tan (90 - \beta)}{\sin (90 - a)}, \qquad \left(-\frac{\pi}{2} < a_\beta < \frac{\pi}{2}\right).$$

These twelve formulas together with the law of sines can be used to solve all oblique spherical triangles on the slide rule shown in the diagram. The rule consists of five scales: A and B are log tan scales; E is an inverted log tan scale; and C and D are log sin scales. A 44-inch classroom demonstration rule gives accuracy corresponding to four-place logarithms for the sides and angles in a normal range.

Example 1. To find the distance between two points whose latitudes and longitudes are known, the formula becomes

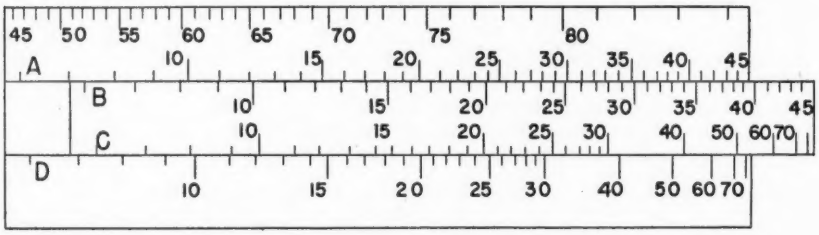
$$\frac{\sin \text{ co-distance}}{\sin \text{ lat } A} = \frac{\sin (\text{co-lat } B + \gamma_a)}{\sin \gamma_a}, \tag{7}$$

where

$$\tan \gamma_a = \frac{\tan \text{ lat } A}{\sin (90 - \text{diff of long})}.$$

Assume A is Seattle (47°36′ $N.$, 122°20′ $W.$) and B is Manila (14°36′ $N.$, 120°57′ $E.$). To compute γ_a, set 47°36′ on scale E opposite $-26°43'$ on scale C and read the auxiliary angle $\gamma_a = -67°40'$ on scale E. Then set 7°44′ on scale C opposite $-67°40'$ on scale D and read the co-distance $= -6°8'$ on scale C opposite 47°36′ on scale D. Hence the distance is 96°8′ or 5768 nautical miles. The distance computed by the use of five-place logarithms is 5769.3 miles.

The four cases involving two like parts and one unlike part can be solved by a similar procedure.



Example 2. Given $a = 78°10'$; $b = 109°20'$; $c = 99°40'$. To compute α_b from (5), set 11°50′ on scale C opposite $-19°20'$ on scale D; move the indicator along until the sum of two numbers on the C and D scales is equal to c or 99°40′. This determines $\alpha_b = -65°25'$ on scale D since $65°25' + 34°15' = 99°40'$. To determine $90 - \alpha$, set $-19°20'$ on scale B opposite $-65°25'$ on scale E and read $(90 - \alpha) = +9°15'$ on scale C opposite the left-hand end of scale D. The other two angles can be found by using the law of sines on the C and D scales. The angle α by the use of five-place logarithms is 80°45.2′ which differs from the slide rule computation by only .2′.

It is obvious that all right spherical triangles can be solved on the slide rule since Napier's rule is reducible to sines and tangents.

By inserting two log N scales, one on the slide between the B and C scales and the other on the base below the D scale, it is possible to solve all cases of plane triangles, although the accuracy is diminished to three places. This follows because the law of sines, the law of tangents, and the law of tangents of the half-angles can be solved on the slide rule as shown in the diagram with the addition of the two log N scales.

A NOTE ON THE PRODUCT OF LINEAR FORMS

RICHARD BELLMAN, University of Wisconsin

G. Pall in the March, 1943 issue of this MONTHLY published an elegant proof of Minkowski's

THEOREM. *If $a_{ij}(i, j=1, 2)$ and b_1, b_2 are real numbers, and if $D=|a_{ij}| \neq 0$, then there exist integers x_i, such that*

$$|(a_{11}x_1 + a_{12}x_2) - b_1| \cdot |(a_{21}x_1 + a_{22}x_2) - b_2| \leqq \frac{|D|}{2}. \tag{1}$$

The conjecture is that for any n, if $L_1, L_2, \cdots, L_n$ are n real homogeneous linear forms in $x_1, x_2, \cdots, x_n$ with determinant D not zero, and $b_1, b_2, \cdots, b_n$ are n real numbers, then there exist integers x_i so that

$$\prod_i |L_i - b_i| \leqq 2^{-n}|D|. \tag{2}$$

It is perhaps interesting to prove that in the very special case where the a_{ij} represent an orthogonal transformation the conjecture is true.

Let $X_1, X_2, \cdots, X_n$ be the solutions of

$$b_i = L_i(X) = \sum_{j=1}^{n} a_{ij}X_j, \qquad i = 1, \cdots, n. \tag{3}$$

Then

$$\prod |L_i(x) - b_i| = \prod_i |L_i(x) - L_i(X)| = \prod_i |L_i(x - X)|. \tag{4}$$

An application of the arithmetic-geometric mean inequality yields

$$\prod_i |L_i(x - X)| \leqq \left[\frac{\sum_1^n (L_i(x - X))^2}{n}\right]^{n/2}. \tag{5}$$

Since the transformation is orthogonal

$$\sum_1^n [L_i(x - X)]^2 = \sum_1^n (x_i - X_i)^2, \tag{6}$$

and so

$$\prod_i |L_i(x - X)| \leqq \left[\frac{\sum_1^n (x_i - X_i)^2}{n}\right]^{n/2}. \tag{7}$$

We can choose the x_i, integers, so that $|x_i - X_i| \leqq \frac{1}{2}$, and then

$$\prod_i |L_i(x - X)| \leqq \frac{1}{2^n}. \tag{8}$$

Since the determinant of an orthogonal transformation is ± 1, this gives the desired result. Orthogonality can be replaced by any condition yielding (6) with the right-hand side greater than or equal to the left.

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 611. *Proposed by D. F. Barrow, University of Georgia*

Suppose n students be standing an examination in a row of seats with an aisle at either end. If they finish in random order, find the average number of disturbances caused by students passing over one another. (If a student passes over 3 others, he causes 3 disturbances. Assume that they go out so as to cause the smallest number of disturbances. Cf. E 531 [1943, 202 and 513].)

E 612. *Proposed by V. Thébault, San Sebastián, Spain*

Show that $1110 \cdot 1111 \cdot 1112 \cdot 1113 = 1235431^2 - 1$ for any radix greater than 5. Generalize this result.

E 613. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Can a triangle have two equal symmedians without being isosceles?

E 614. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Prove that the polynomial $(x+y)^n - x^n - y^n$ is divisible by x^2+xy+y^2 when $n \equiv 5 \pmod 6$, and by $(x^2+xy+y^2)^2$ when $n \equiv 1 \pmod 6$.

E 615. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be real numbers such that $0 < g_p < 1$ and such that the series

$$g_1 + (1 - g_1)g_2 + (1 - g_2)g_3 + \cdots$$

converges to a sum not exceeding unity. Establish the convergence of

$$1+\sum_{p=1}^{\infty}\frac{g_1g_2\cdots g_p}{(1-g_1)(1-g_2)\cdots(1-g_p)}.$$

SOLUTIONS

Sums of Residues of Powers

E 575 [1943, 326]. *Proposed by P. S. Donchian, Hartford, Connecticut*

Let (n^r) denote the standard residue of n^r modulo p (an odd prime), so that $(n^r)\equiv n^r \pmod p$ and $0\leqq(n^r)<p$. Prove that the $p-2$ numbers

$$\sum_{n=1}^{p-1}(n^r) \qquad (r=1,\cdots,p-2)$$

are a permutation of the $p-2$ numbers

$$\sum_{r=0}^{p-2}(n^r) \qquad (n=2,\cdots,p-1),$$

and that all these numbers are divisible by p.

Solution by E. P. Starke, Rutgers University. Let d be a primitive root modulo p. This implies that the numbers

$$d,\ (d^2),\ \cdots,\ (d^{p-1})=d^0=1$$

are a permutation of $1, 2, \cdots, p-1$. Let us put $n=d^g$ $(0\leqq g\leqq p-2)$.

Consider a particular value of r, $1\leqq r\leqq p-2$; suppose $(r, p-1)$, the greatest common divisor of r and $p-1$, is equal to f; and put $u=r/f$, $v=(p-1)/f$, so that $v>1$. Since

$$(d^{gu f})^v\equiv(d^{p-1})^{gu}\equiv 1,$$

we see that $(d^r)^g$ is a root of the congruence

$$x^v\equiv 1 \pmod p \tag{1}$$

for any choice of g. Further, (1) has just v incongruent roots, namely

$$1,\ (d^r),\ (d^{2r}),\ \cdots,\ (d^{(v-1)r}). \tag{2}$$

Thus there are only v distinct values possible for (d^{rg}). Each is assumed f times; for, the relation

$$(d^r)^{g+v}\equiv d^{rg}d^{rv}\equiv d^{rg}d^{(p-1)u}\equiv(d^r)^g$$

implies that the f numbers

$$(d^r)^g,\ (d^r)^{g+v},\ (d^r)^{g+2v},\ \cdots,\ (d^r)^{g+(f-1)v}$$

are congruent each to each. (In this and similar results, any exponent not less than $p-1$ should be diminished by $p-1$.) Hence the $p-1=vf$ numbers (d^{rg}) may be separated into f sets each consisting of the v numbers (2). Hence

$$\sum_{n=1}^{p-1}(n^r) = \sum_{g=0}^{p-2}(d^{rg}) = f\{1 + (d^r) + (d^{2r}) + \cdots + (d^{(v-1)r})\} \qquad (r = 1, 2, \cdots, p-2). \tag{3}$$

Corresponding to each f it is known that there are $\phi(v)$ distinct values of r such that $(r, vf)=f$. Thus, of the $p-2$ sums (3), each occurs $\phi(v)$ times.

Consider now a particular value of $n=d^g(1\leqq g\leqq p-2)$; suppose $(g, p-1)=k$; and put $s=(p-1)/k$, $t=g/k$. The relation

$$(n^{r+s}) \equiv d^{gr+sg} \equiv d^{gr}d^{(p-1)t} \equiv d^{gr} \equiv (n^r)$$

implies that the k numbers

$$(n^r), (n^{r+s}), (n^{r+2s}), \cdots, (n^{r+(k-1)s})$$

are congruent each to each. Hence the $p-1=ks$ numbers

$$n^0 = 1, n = (d^g), (n^2) = (d^{2g}), \cdots, (n^{p-2}) = (d^{(p-2)g})$$

separate into k sets, each set consisting of the s numbers

$$1, (d^g), (d^{2g}), \cdots, (d^{(s-1)g})$$

which, as in (2), are the incongruent roots of the congruence

$$x^s \equiv 1 (\bmod p), \tag{4}$$

where $s>1$. Thus we have

$$\sum_{r=0}^{p-2}(n^r) = k\{1 + (d^g) + (d^{2g}) + \cdots + (d^{(s-1)g})\} \qquad (g = 1, 2, \cdots, p-2). \tag{5}$$

Corresponding to each k, there are $\phi(s)$ distinct values of g, and hence of n, such that $(g, ks)=k$. Thus, of the $p-2$ sums (5), each occurs $\phi(s)$ times.

If now k is taken equal to f, so that $s=v$, the congruences (1) and (4) are identical and have identical sets of roots, so that the sums (3) and (5) are identical. Furthermore each occurs the same number $\phi(s)=\phi(v)$ of times. Finally, the sum of all roots (2) of congruence (1) is congruent to minus the coefficient of x^{v-1}. Since this is zero, each of the sums (3) and (5) is divisible by p. Since $g=1, 2, \cdots, p-2$ correspond in some order to $n=2, 3, \cdots, p-1$, the proof is complete.

Seven Consecutive Digits forming a Cube

E 578 [1943, 386]. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect cube whose digits form a permutation of consecutive digits. (Cf. E 538.)

Solution by Frank Hawthorne, Allegheny College. Since the number is restricted to ten digits, the root cannot exceed 2154. Roots ending in 0, 14, 42, 53, 64, 71, 77, 92, and 99 may be eliminated because the cubes end in repeated digits. If the number contains fewer than ten digits, the root cannot end in 69; if fewer than nine digits, in 02, 31, 39, 52; if fewer than eight digits, in 09, 43, 48, 61, 98; for in each case the last two digits of the cube differ too greatly. Examining the remaining numbers with the aid of a table, we find that the only perfect cube satisfying the condition is

$$8365427 = 203^3.$$

The only higher powers satisfying the same condition are the fifth powers 32 and 243.

Also solved by W. E. Buker, M. L. Constable, Howard Eves, Irving Kaplansky, E. P. Starke, and the proposer.

Submatrices of a Matrix

E 579 [1943, 386]. *Proposed by Howard Eves, Syracuse University*

Show that a matrix of m rows and n columns contains $(2^m-1)(2^n-1)$ submatrices.

Solution by Alan Wayne, Flushing, L. I. The number of different combinations of rows from m rows is

$$\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{m}=2^m-1.$$

Similarly we can choose 2^n-1 different sets of columns. Hence we get

$$(2^m-1)(2^n-1)$$

selections of rows and columns, or submatrices.

Also solved by W. E. Buker, Orrin Frink, Jr., Irving Kaplansky, E. P. Starke, C. W. Topp, and the proposer. The proposer remarks that the problem was suggested by exercise 3 on page 22 of Albert's *Introduction to Algebraic Theories*, where the reader is asked to write out all the submatrices of a given 4×5 matrix—a task of listing $15\cdot31=465$ submatrices!

A Hot Water Tank

E 580 [1943, 386]. *Proposed by Albert Furman, Infantry School, Fort Benning*

A closed tank containing V gallons of water at temperature T has an inlet pipe which supplies water at temperature t. Assuming an ideal situation where there is no loss of heat and an instantaneous diffusion in the mixture, show that the temperature of v gallons of water drawn from the tank into an open container is

$$t+(T-t)(1-e^{-v/V})V/v.$$

Solution by O. J. Karst, Newark College of Engineering. Let q be the temperature of the water in the closed tank after the passage of p gallons through the

system. The physical set-up of the problem leads to the differential equation

$$V dq = (t - q)dp$$

with the initial condition that $q = T$ when $p = 0$. The solution is

$$q = t + (T - t)e^{-p/V}.$$

The temperature of the water in the open container is obtained as the weighted average of the temperatures of the elements of water which flow into it, the temperature q of an element being weighted by its volume dp. After the passage of v gallons, this weighted average is

$$\int_0^v q dp \Big/ \int_0^v dp = \int_0^v \{t + (T - t)e^{-p/V}\} dp/v$$
$$= t + (T - t)(1 - e^{-v/V})V/v.$$

Also solved by R. K. Allen, W. B. Campbell, Howard Eves, C. W. Johnson, S. Karlin, E. P. Starke, and the proposer.

The Farmer's Two Sons

E 581 [1943, 386]. *Proposed by D. K. Kazarinoff, University of Michigan*

A farmer died and left his two sons a herd of cattle which they sold. The number of dollars received per head was the same as the number of heads. With the proceeds of the sale the sons bought sheep at \$10 each and one lamb for less than \$10. The sheep and the lamb were divided between the two brothers so that each received the same number of animals. How much should the son who received only sheep pay to the son who received the lamb in order that the division should be equitable?

I. *Solution by Irving Kaplansky, Harvard University.* If x is the number of cattle, y the number of sheep, z the price of the lamb, we have $x^2 = 10y + z$, with y odd and $z < 10$. But it is readily shown that the penultimate digit of a square is odd if and only if the last digit is 6. Thus $z = 6$, and the luckier son should hand over \$2.

II. *Solution by L. R. Ford, Illinois Institute of Technology.* Let $10n$ be the multiple of 10 nearest the total number of cows. Then the sum available for the purchase of sheep is

$$(10n + p)^2 = 100n^2 + 20np + p^2,$$

where $p = 0, \pm 1, \pm 2, \pm 3, \pm 4$, or ± 5. Now $100n^2 + 20np$ will purchase an even number of sheep, which can be divided. The sum that remains is

$$p^2 = 0, 1, 4, 9, 16 \text{ or } 25.$$

The case $p^2 = 16$ will permit the purchase of a sheep and a lamb, while the other cases are impossible. The recipient of the \$6 lamb should be paid \$2 by his brother.

Also solved by R. K. Allen, D. H. Browne, W. E. Buker, M. L. Constable, A. H. Copeland, J. S. Cromelin, M. A. Dernham, William Douglas, C. W. Emmons, Howard Eves, N. G. Gunderson, F. A. Haight, Frank Hawthorne, Norbert Kaufman, Victoria Kloch, Elmer Latshaw, Walter Penney, W. C. Rufus, E. D. Schell, Jesse Silverman, E. P. Starke, Michael Wilensky, and C. H. Wolfe. Starke remarks that a duplicate of this problem (except that each sheep cost \$12) appeared fourteen years ago as No. 3379 [1930, 162].

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4111. *Proposed by G. Pólya, Stanford University*

The axes of two unlimited circular cylinders intersect. Let a and b be the radii of the cylinders, $a>b$; γ the angle between the axes; and V the volume of the intersection (the volume within both cylinders). Writing $b/a=k$, prove that

$$\frac{16ab^2}{3\sin\gamma} < V < \frac{2\pi ab^2}{\sin\gamma},$$

$$\frac{2\pi ab^2}{\sin\gamma}\left[1-\left(1-\frac{8}{3\pi}\right)k^2\right] < V < \frac{16ab^2}{3\sin\gamma}\,\frac{5-k^2}{4},$$

and generally for $n=0, 1, 2, 3, \cdots$

$$0 < \frac{2\pi ab^2}{\sin\gamma}\left[1-\sum_{\nu=1}^{n}\left(\frac{1}{2}\,\frac{3}{4}\cdots\frac{2\nu-1}{2\nu}\right)^2\frac{k^{2\nu}}{(\nu+1)(2\nu-1)}\right]-V$$

$$< \frac{2\pi ab^2}{\sin\gamma}\left[1-\frac{8}{2\pi}-\sum_{\nu=1}^{n}\left(\frac{1}{2}\,\frac{3}{4}\cdots\frac{2\nu-1}{2\nu}\right)^2\frac{1}{(\nu+1)(2\nu-1)}\right]k^{2n+2}.$$

Note. The proposer is indebted to F. R. Morris for the problem of finding approximations to V.

4112. *Proposed by N. A. Court, University of Oklahoma*

If three rulers, chosen arbitrarily, of the same system of a given hyperboloid are taken for the edges of a parallelopiped, the diagonals of the parallelopiped meet in a fixed point.

4113. *Proposed by J. H. Butchart, Grinnell College*

If every member of one family of developables cuts every member of another family of developable surfaces orthogonally, the edges of regression of the two families form a system of orthogonal geodesics on a single developable surface.

4114. *Proposed by V. Thébault, San Sebastián, Spain*

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are taken each once and only once to form three numbers N, n, r so that $N=n^3-r$. Find these three numbers.

SOLUTIONS

Euler Line

4057 [1942, 616]. *Proposed by J. R. Musselman, Western Reserve University*

Let B_1, B_2, B_3 be the points symmetric to the vertices of triangle $A_1A_2A_3$ in its circumcenter O, and let C_1, C_2, C_3 be the reflections of A_i in the perpendicular bisector of the sides of $A_1A_2A_3$. It is known that the circles OB_1C_1, OB_2C_2, OB_3C_3 meet at a point P. Show that P lies on the Euler line of $A_1A_2A_3$ and that O is the midpoint of PD, where D is the inverse in the circumcircle of the orthocenter H of $A_1A_2A_3$.

Solution by Hüseyin Demir, Columbia University. Let $G_1G_2G_3$ be the triangle formed by the straight lines A_iC_i so that $A_1A_2A_3$ is its medial triangle, the circumcircle (O) of the latter is its ninepoint circle, G_iC_i are its altitudes, its orthocenter H' is the symmetric of H with respect to O. and the straight lines C_iB_i are concurrent in H'. Let P be the point where the circle (OB_1C_1) cuts OH', *i.e.*, OH. We have $H'O\cdot H'P=H'C_1\cdot H'B_1=H'C_1\cdot H'B_i$; hence the circles (OB_iC_i) intersect again in P. The inverse of (OB_1C_1) with respect to (O) is B_1C_1, and hence $OH'\cdot OP=\overline{OC}_1^2=R^2$. Since $OH\cdot OD=R^2$ and $OH=H'O$, we must have $OD=PO$.

Solved also by H. Eves using inversion with respect to O and power $-R^2$ which gives a concise proof.

Tetrahedron, Spheres, Sums of Powers

4064 [1942, 689]. *Proposed by V. Thébault, San Sebastián, Spain*

Given a tetrahedron $ABCD$: (1) Find the locus of points M such that the sum of the powers of the vertex A with respect to the spheres with diameters MB, MC, MD is constant. (2) Find the point M such that for the spheres with the diameters MA, MB, MC, MD the sum of the powers of a vertex with respect to the three spheres not passing through that vertex is the same for the four vertices. Show that the point M in this case is the symmetric of the centroid with respect to the circumcenter of the tetrahedron.

Solution by Howard Eves, Syracuse University. (1) We use vector algebra, denoting the points A, B, C, D, M by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{m}$. The sum of the powers of A with respect to the spheres on MB, MC, MD as diameters is

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{m})+(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{m})+(\mathbf{a}-\mathbf{d})\cdot(\mathbf{a}-\mathbf{m}).$$

Therefore we have

$$(3\mathbf{a} - \mathbf{b} - \mathbf{c} - \mathbf{d})\cdot(\mathbf{a} - \mathbf{m}) = k, \text{ a constant,}$$

or, setting $\mathbf{g} = \frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$, (for the centroid G),

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{a} - \mathbf{m}) = k/4.$$

This last result shows that the locus of M is a plane perpendicular to AG.

(2) Now take the circumcenter as origin, so that $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = \mathbf{d}\cdot\mathbf{d} = R^2$, where R is the circumradius. Then, for some k, we have

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{a} - \mathbf{m}) = (\mathbf{b} - \mathbf{g})\cdot(\mathbf{b} - \mathbf{m}) = (\mathbf{c} - \mathbf{g})\cdot(\mathbf{c} - \mathbf{m})$$
$$= (\mathbf{d} - \mathbf{g})\cdot(\mathbf{d} - \mathbf{m}) = k/4.$$

Adding we have

$$(\mathbf{a} - \mathbf{g})\cdot\mathbf{a} + (\mathbf{b} - \mathbf{g})\cdot\mathbf{b} + (\mathbf{c} - \mathbf{g})\cdot\mathbf{c} + (\mathbf{d} - \mathbf{g})\cdot\mathbf{d} = k, \quad \text{or} \quad 4R^2 - \mathbf{g}\cdot\mathbf{g} = k,$$

whence

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{a} - \mathbf{m}) = R^2 - \mathbf{g}\cdot\mathbf{g}.$$

Expanding and collecting we see that

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = 0.$$

Since we similarly have

$$(\mathbf{b} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = (\mathbf{c} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = (\mathbf{d} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = 0,$$

it follows that $\mathbf{m} = -\mathbf{g}$, or M is the symmetric of G in O. We have incidentally shown that the common power is $k = 4(R^2 - OG^2)$.

Solved also by P. D. Thomas.

Three Point Collinearity

4065 [1943, 65]. *Proposed by P. Erdös, Princeton, N. J.*

(1) Let n given points have the property that the straight line joining any two of them passes through a third point of the set. Show that the n points lie on a straight line.

(2) Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines.

Solution by Robert Steinberg, Student, University of Toronto. (1) We shall suppose that the n points do not all lie on one line, and show that this leads to a contradiction. Let A, B, C be three noncollinear points among the n, and let a be any line through A, in the plane ABC, which does not contain any other point of the set. Let those joins of pairs of the n points which meet a do so at points $P_1, P_2, \cdots$. There will be at least one of the points $P_1, P_2, \cdots$ which does not coincide with A, *viz.* the one that lies on BC; thus the points A, P_1,

$P_2, \cdots$ decompose the line a into two or more segments. Hence we can find a point P among $P_1, P_2, \cdots$, such that one of the two segments AP contains none of the points $P_1, P_2, \cdots$ within it.

By our hypotheses, P is not one of the set of n points, but lies on a line containing at least three of them. Let these three points be named Q, R, S in such an order that P is separated from R by Q and S. Since A and R are two points of the set, the line AR will contain a third point of the set, say O. Let a meet QO at P_1, and SO at P_2. Then, by perspectivity from O, P_1 and P_2 separate P from A; *i.e.*, the points P_1 and P_2 lie *one* in *each* of the two segments AP. Thus we come to a contradiction; so in fact all the n points must be collinear.

(2) Now, suppose we have n points not all collinear, and let m be the number of distinct joins of pairs of these points. By the theorem just proved, at least one of these m lines contains only two points of the set, say A_1 and A_2. Consider the set of $n-1$ points obtained by excluding A_1. This set will contain at most $m-1$ distinct joins of pairs of points, as the line A_1A_2 is not one of these joins. We can repeat this process, excluding one point from the set in the manner described, as long as the remaining points are not all collinear. Suppose that after r such steps the remaining $n-r$ points are all collinear. Then, before the rth step we had $n-r+1$ points, of which just $n-r$ were collinear. Thus there were exactly $n-r+1$ distinct joins. But, since $r-1$ points have been subtracted from the original set, we have

$$n - r + 1 \leqq m - (r - 1),$$

i.e.,

$$n \leqq m.$$

Thus there are at least n distinct joins.

Solved also by R. C. Buck, T. Grünwald and N. E. Steenrod.

Editorial Note. The proposer enclosed with the problem an outline of Grünwald's solution of part (1) and remarked that part (2) could be easily proved by induction using (1). This proof of (1) may be put in the following form. The assumption that the n points do not lie on a straight line leads to a contradiction as follows.

Consider only those points A_i which lie in a plane determined by three non-collinear points, and project them into the points B_i, from a center O not in the plane, on a second plane parallel to OA_1 but not to any other OA_i. In the second plane we have a system of parallels each containing at least three points including the point at infinity B_1. Let l be the join of two points meeting the system at the least positive angle. Then on l there are three finite points B_i, B_j, B_k, where we suppose that B_j lies between B_i and B_k, and there are three parallels through these points. On the parallel through B_j there is at least one finite point B' distinct from B_j, and we now have the contradiction that one of the two joins B_iB', B_kB' must make a smaller positive angle with the parallels. Hence the original n points lie on a single straight line.

The proof by Buck of part (1) is similar to Grünwald's proof, and he stated that this part of his proof is not original; he was uncertain about the origin of the earlier proof. Steenrod likewise reduced part (1) to its two dimensional case; he then derived the dual theorem in the projective plane (where we have n distinct lines such that the intersection of any two is on a third) from the impossibility of constructing a map whose regions, occurring six or more at each vertex, have each three or more sides.

L. M. Kelly remarked that part (1) was proposed by Sylvester in the *Educational Times*, Mathematical Question 11851, vol. 59, 1893, p. 98; but it was not satisfactorily solved there. Erdös stated that, at Oslo, Karamata asked him about this problem which he had seen stated without proof in an old book about mechanics.

Multiplication by Addition and Reciprocation

4062 [1942, 689]. *Proposed by N. S. Mendelsohn, University of Toronto*

Show how the operation of multiplying by a real number may be expressed in terms of the operations of adding a real number and of reciprocating.

Solution by the Proposer. Multiplication by a negative number $-n^2$:

$$-n^2x = n + \cfrac{1}{-\cfrac{1}{n} + \cfrac{1}{n + \cfrac{1}{x}}}.$$

Multiplication by a positive number n^2:

$$n^2x = n + \cfrac{1}{-\cfrac{n+1}{n} + \cfrac{1}{1 + \cfrac{1}{-(n+1) + \cfrac{1}{x}}}}.$$

Self Polar Triangle Inscribed in Another Conic

4063 [1942, 689]. *Proposed by H. S. M. Coxeter, University of Toronto*

In projective geometry the porism of triangles inscribed in one conic and self-polar for another is commonly proved by showing that if one such triangle exists, we can find another with one vertex at *any* given point on the first conic. This statement is easily seen to be valid in complex geometry. Discuss its possible failure in real geometry.

Partial Solution by the Prosposer. Consider the conics $yz+zx+xy=0$ and $x^2+y^2=z^2$. The polar of $(2, 2, -1)$ for the latter fails to meet the former.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Flora Dinkines of the University of South Carolina has been promoted to an adjunct professorship.

Edwin Eagle of Taft Junior College, California, has been appointed to an assistant professorship at Oregon State College.

Professor J. M. Howie of Nebraska Wesleyan University has retired.

Assistant Professor W. J. Kirkham of Oregon State College is on leave of absence. He holds the rank of lieutenant in the United States Navy.

Associate Professor A. E. Meder, Jr., of New Jersey College for Women has been appointed secretary of Rutgers University.

Professor Richard Morris will retire as head of the department of mathematics of the New Jersey College for Women on June 30, 1944. Associate Professor C. A. Nelson will then become head of the department. Professor Morris will continue to teach in Rutger's University.

Associate Professor W. S. Slauch of New York University has been appointed professor emeritus.

Associate Professor Raleigh Schorling of the University High School of the University of Michigan has been appointed to a professorship in the department of education at the University of Michigan.

Professor E. R. Smith of the Florida State College for Women has retired.

S. Grace Smyth of Knox College has been promoted to an associate professorship.

Assistant Professor G. J. Stigler of the University of Minnesota has been promoted to an associate professorship in economics.

Fern Welker of the University of Toledo has been promoted to an assistant professorship.

Associate Professor Marie Johnson Yeaton of Oberlin College has resigned.

The following appointments to instructorships are announced:

Oregon State College: Florence Bakkum, L. R. Foote, Fred Young
Syracuse War Training Program at Auburn: Dr. S. Helen Taylor
University of Pennsylvania Physics Department: G. H. Wilson

Associate Professor G. A. Pfeiffer of Columbia University died on January 4, 1944.

Professor H. L. Rietz of the State University of Iowa died on December 7, 1943. He was a charter member of the Mathematical Association.

The Editor-in-Chief wishes to express his indebtedness to the following persons who have served as referees of papers during 1943.

R. P. Agnew, A. A. Albert, Emil Artin, E. F. Beckenbach, L. M. Blumenthal, J. W. Bradshaw, Louis Brand, H. E. Bray, R. W. Brink, H. E. Buchanan, W. B. Carver, W. B. Caton, R. V. Churchill, C. J. Coe, N. B. Conkwright, A. H. Copeland, H. S. M. Coxeter, D. R. Curtiss, H. T. Davis, W. M. Davis, John DeCicco, Max Dehn, J. S. Frame, Orrin Frink, Jr., M. G. Gaba, Edward Helly, M. R. Hestenes, T. J. Higgins, Dunham Jackson, R. A. Johnson, A. J. Kempner, Mayme I. Logsdon, C. C. MacDuffee, S. B. Meech, E. J. Moulton, J. R. Musselman, C. V. Newsom, W. C. Randels, Haim Reingold, M. A. Sadowsky, Peter Scherk, O. F. G. Schilling, Harold Shapiro, I. M. Sheffer, G. E. F. Sherwood, N. E. Steenrod, E. B. Stouffer, T. Y. Thomas, G. E. Wahlin, L. E. Ward, Marie J. Weiss, L. R. Wilcox, F. E. Wood, and R. C. Yates.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE REPUBLICATION OF FOREIGN MATHEMATICAL TABLES

The Office of Alien Property Custodian has licensed, during the past several months, the reprinting of scientific and technical books, of enemy origin, which are not available in a quantity sufficient to meet the demands of the wartime operations of science and industry.

In this connection the custodian has received several queries concerning the possibility of licensing the republication of additional Mathematical Tables. Licensed for republication and now available for purchase are Jahnke and Emde, "Funktionentafeln mit Formeln und Kurven," 1938; Jean Peters, "Siebenstellige Werte der Trigonometrischen Funktionen," 1938, and his "Achstellige Tafel," 1939.

Before a definite decision can be made regarding the licensing of additional Mathematical Tables for republication, it is necessary for the custodian to be informed about the extent of the need of such tables and to receive suggestions of specific titles for consideration. This can be accomplished if suggestions of specific significant tables are sent by individuals to the Office of Alien Property Custodian, Washington, D. C. These suggestions or any inquiries should be addressed to the undersigned.

HOWLAND H. SARGEANT,
Chief, Division of Patent Administration
Office of Alien Property Custodian,
Washington, D. C.

QUOTAS IN MILITARY TRAINING PROGRAMS

The American Council on Education has made a study of the number of military personnel receiving instruction in the Army and the Navy College Training Programs. The data indicate that on October 1, 1943, there were 276,821 men in the four major programs. The distribution follows.

	Number of Institutions	*Number of Trainees*
Army Specialized Training	216	129,080
Army Air Forces	151	66,512
Navy College Training	244	73,486
Navy Air Force	17	7,743

When duplications are eliminated from the total of institutions, it is found that there are 440 participating colleges and universities.

Since the compilation of the figures above, there has been a slight decrease in the total quota allotted to the Army Specialized Training Program. In fact, the rumor has been circulated that the Program would shortly be liquidated. Consequently, upon December 13, the War Department released the following statement.

"In answer to inquiries, the War Department announced today that the Army Specialized Training Program is not in process of liquidation. The number of soldiers in the Program will depend in the future, as in the past, on the actual needs of the Arms and Services.

"In this connection the Secretary of War said: 'The number of soldiers assigned for training under the ASTP will be changed from time to time so as to accord with the needs of the Army and the available manpower. It is now being reduced—but may later be either increased or still further reduced as the exigencies of the military situation or military training make advisable.'"

At the present time no essential change is contemplated in the total Navy quota. It is interesting to note, however, that the number of men from active duty being returned for study in the Navy V-12 Program has recently been doubled.

CENTRAL CLEARING AGENCY FOR ACCREDITATION

The following memorandum was issued from the Headquarters of the United States Armed Forces Institute, Madison 3, Wisconsin.

1. The United States Armed Forces Institute (originally the Army Institute) was established on 1 April, 1942. The Institute offers to men and women in the Armed Services formal off-duty correspondence instruction as an opportunity to increase their military efficiency, to prepare for a civilian occupation after the war, or to complete requirements for a high school diploma or college degree.

2. The Institute has established a Central Clearing Agency for Accreditation to help service personnel secure academic credit for their in-service training and experience. The following regional accrediting associations have endorsed the procedure of the Central Clearing Agency for Accreditation:

The New England Association of Colleges and Secondary Schools
The North Central Association of Colleges and Secondary Schools
The Middle States Association of Colleges and Secondary Schools
The Southern Association of Colleges and Secondary Schools
The Northwest Association of Secondary and Higher Schools

3. The Institute does not grant nor recommend academic credit, but will assemble all available information concerning the in-service training and experience of service personnel and forward a complete, official report of the data collected to the designated high school or college for evaluation.

4. The Institute will also assemble an official report and forward it to a prospective employer to assist the serviceman in his future civilian job plans. This report, however, will be furnished only to service personnel who are about to receive their discharge from the armed forces.

5. In order to initiate the accreditation service, it is necessary that each serviceman secure and fill out the application form "Request for Report of Educational Achievement." This form can be secured by writing to the Institute. It is suggested that high schools and colleges advise any of their former students now in the Armed Forces who desire an evaluation of their in-service training and experience to write to the Institute requesting this application form.

CARL W. HANSEN,
Lt. Col., A.G.D., Commandant

ENLISTED MEN ENTERING THE NAVY V-12 PROGRAM

Due to general dissatisfaction with the procedure followed by the Navy in selecting enlisted men for the V-12 Program, more rigorous academic criteria have recently been instituted. No complete announcement of the new standards has been officially released for publication, but available reports indicate that the following academic qualifications are now necessary.

1. Enlisted men will not be recommended for the V-12 Program unless they have had a minimum of two years of high school mathematics. Preference will be given to those who have had algebra, geometry, and trigonometry.

2. All enlisted men entering the Program must have passed a general classification test with a high score. In the case of candidates outside the continental United States who may have had no opportunity to take the classification test, an oral examination over algebra, geometry, English, and history will be given to determine whether the men are qualified for college work. Men satisfactorily passing the oral examination must still be given the written test upon their arrival at Navy stations designated as assembly points.

It is the present intention also to give enlisted men gathered at the assembly points some review work in mathematics before assignment to institutions participating in the V-12 Program.

THE NAVY V-5 PROGRAM

The Navy V-5 Program is for the purpose of training aviation cadets for the Navy Air Force. Candidates selected by the Naval Aviation Cadet Selection Boards follow the training schedule given below.

First three months—Naval Flight Preparatory School

No flight training. The entire time is spent in ground school on the following subjects: review of mathematical processes, practical arithmetic, trigonometry and geometry, fundamentals of physics, aerial navigation using plotting boards, aerology, aircraft structures, aircraft engines, Naval indoctrination, radio code, aircraft and ship recognition, physical training and drill.

Second three months—CAA-WTS Flight Training School

Initial stage of flight training with a maximum of sixty hours of flight. The ground school consists of a review of navigation, review of aerology, review of aircraft structures and engines, continued practice in radio code and semaphore, recognition, civil air regulations, physical training and drill.

Third three months—Naval Pre-Flight School

No flight training. The emphasis is on physical training and bodily contact sports. The ground school consists of a review of navigation with an introduction to celestial navigation, aerology, code, semaphore, and blinker, recognition, and Naval history.

Next three months or less—Naval Primary Flight Training Base

Advanced flight training with formation and night flying. The ground school is secondary and consists of further study of navigation, aerology, recognition, code and blinker, physical training and drill.

Next three to four months—Naval Air Station

Final transition to combat types of aircraft with emphasis on combat tactics, aerial gunnery, instrument flying, and precision air work. The ground school consists of precision work in navigation, aerology, recognition, instrument flying, aerial gunnery, and combat tactics. Information on other courses is restricted. At the end of this phase of the training the cadets are commissioned as ensigns in the USNR.

Next one to two months—Transition Training Base Naval Air Station

Transition to the type of plane to be used in combat, and the organization of combat squadrons. The ground school is of an individualized type having to do with individual squadron problems and acquainting the new pilots with combat routine.

Many aviation candidates are now taking the first two semesters of the Navy V-12 Program, being designated V-12A. At the end of this period of training, they will go to a Naval Flight Preparatory School and start the regular program as outlined above.

The mathematics now being offered in the Naval Flight Preparatory Schools is taught for four weeks, twelve hours per week. The outline of the course with the hours allotted to each topic is given below.

First week: Fundamental operations and conversion factors (4), fractions (3), percentage (1), review (1), test (1), powers and roots (2).

Second week: Powers and roots (3), angular measurement (2), vectors (3), review (1), test (1), equations (2).

Third week: Equations (4), variation (4), graphs (2), review (1), test (1).

Fourth week: Geometry (4), trigonometry (4), test (1), review of entire course (2), final examination (1).

THE MATHEMATICAL ASSOCIATION OF AMERICA

REPORT OF THE TREASURER FOR THE YEAR 1943

The following report of the Secretary-Treasurer as Treasurer for the year 1943 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on January 15, 1943

(See Cairns' report, pp. 208–209 of the Monthly for March, 1943)

Current Fund (checking account)		$ 1,172.47
Savings Account, Cleveland Trust Company		1,713.66
Savings Account, Oberlin Savings Bank		811.20
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 7,144.41	
Chace Fund	8,499.04	
Houck Fund	8,405.98	
Chauvenet Fund	614.40	
Life Membership Fund	824.30	
General Fund	20,438.19	45,926.32
		$49,623.65

II. Current Fund Account for 1943

Receipts		Expenditures	
Balance, Jan. 15, 1943	$1,172.47	Monthly	
Individual dues	8,650.80	Publications	$6,299.83
Institutional dues	799.90	Reprints	59.89
Initiation fees	204.00	Editor-in-Chief's office	278.65
Subscriptions	1,922.33	Secretary-Treasurer's office	
Sale of back numbers Monthly	101.95	Clerical help	3,428.88
Advertising	518.00	Printing	270.74
Sale of Archibald's Outline	72.50	Postage	574.93
Sale of the library	1,661.64	Supplies, etc.	381.79
Interest on General Fund	538.44	"Smith list"	101.82
Sale of Carus Monographs	1,036.25	Bank charges	93.93
Interest on Carus Fund	188.19	President's office	103.23
Income from Hardy Fund	120.00	Exec. and Finance committees	82.85
Sale of Papyrus (for Chace Fund)	40.00	Regional Governors	108.93
Interest on Chace Fund	223.90	Subventions	
Interest on Chauvenet Fund	16.19	A.M.S.	100.00
Interest on Houck Fund	221.44	Mathematical Reviews	500.00
Interest on Life Membership Fund	21.71	Duke Journal (from 1942)	50.00
Transferred from Carus Fund	305.14	Expense of Sections	132.72
Transferred from Houck Fund	178.56	Meetings	230.21
Transferred from Life Membership Fund	88.96	Purchase of back numbers	68.64
		To B. F. Finkel (Hardy Fund)	120.00
		Subscriptions to Annals	7.50
		Expense, sale of library	60.32
		Paid out for Carus Fund	1,529.58
		Paid out for Houck Fund	400.00
		To Clev. Trust Co. for General Fund	262.80
		Transferred to Chace Fund	263.90
		Transferred to Chauvenet Fund	16.19
		Transferred to Life Membership Fund	21.71
		Transferred to General Fund	270.86
		Balance, Dec. 31, 1943	2,262.47
	$18,082.37		$18,082.37

III. Savings Account, Cleveland Trust Company

Balance, Jan. 15, 1943	$1,713.66		
Interest (half-year)	8.31	Balance, Dec. 31, 1943	$1,721.97
	$1,721.97		$1,721.97

IV. Invested Funds, Cleveland Trust Company

Cash balance, Jan. 15, 1943.....	$ 27.82		
Market Value of Securities, Jan. 15, 1943.....	45,898.50		
Increase in value of securities....	1,140.88	Market value of securities, Dec. 31, 1943.....	$47,330.00
Cash from Current Fund.....	262.80		
	$47,330.00		$47,330.00

List of Securities

	Par Value	Market Value Dec. 31, 1943
U. S. Savings Bonds.....	$1,700.00	$ 1,550.00
U. S. Treasury Note, 1%, Ser. A, 1946.....	3,000.00	3,000.00
U. S. Treasury Bond, 2¼%, 1947.....	1,000.00	1,030.00
U. S. Treasury Bonds, 2%, 1950.....	3,000.00	3,060.00
U. S. Treasury Bonds, 1¾%, 1948.....	2,000.00	2,020.00
U. S. Savings Bonds, 2½%, Ser. G, 1953.....	3,000.00	3,000.00
U. S. Savings Bonds, 2½%, Ser. G, 1954.....	8,200.00	8,200.00
Canadian Nat. Ry. Co. Bonds, 4½%, 1956.....	2,000.00	2,300.00
Gatineau Power Co. 1st Mort. Bonds, 3¼%, Ser. A, 1969.....	2,000.00	1,960.00
Shawinigan W. and P. Co. 1st Mort. Bonds, 4¼%, 1970.....	2,000.00	2,080.00
C. and O. Ry. Co. Ref. Mort. Bonds, 3½%, Ser. D. 1996.....	3,000.00	3,210.00
Penn. R. R. Co. Gen. Mort. Bonds, 3¼%, Ser. C, 1970.....	2,000.00	1,980.00
Amer. Tel. and Tel. Co. Conv. Deb. Bonds, 3%, 1956.....	2,000.00	2,320.00
Columbus and So. Ohio Elec. Co. 1st Mort. Bonds, 3¼%, 1970...	2,000.00	2,180.00
Commonwealth Edison Co. Conv. Deb. Bonds, 3½%, 1958.....	2,000.00	2,240.00
Montana Power Co. 1st Ref. Mort. Bonds, 3¾%, 1966.....	3,000.00	3,180.00
New York Steam Corp. 1st Mort. Bond, 3½%, 1963.....	1,000.00	1,070.00
Texas Power and Light Co. 1st Mort. Bond, 5%, 1956.....	1,000.00	1,080.00
Phelps Dodge Corp. Conv. Deb. Bond, 3½%, 1952.....	1,000.00	1,070.00
Hotel Cleveland Site, Land Trust Certifs.....	700.00	800,00
		$47,330.00

V. Carus Fund

Balance, Jan. 15, 1943.....	$7,144.41	Publication of 7th Monograph...	$1,224.58
Sale of Monographs.....	1,036.25	Honorarium, 7th Monograph.....	300.00
Interest.....	188.19	Expenses of committee.....	5.00
Increase in value of securities....	177.46	Balance, Dec. 31, 1943.....	7,016.73
	$8,546.31		$8,546.31

VI. Chace Fund

Balance, Jan. 15, 1943.....	$8,499.04		
Sale of Papyrus.....	40.00		
Interest.....	223.90		
Increase in value of securities....	211.13	Balance, Dec. 31, 1943.....	$8,974.07
	$8,974.07		$8,974.07

VII. Houck Fund

Balance, Jan. 15, 1943	$8,405.98		
Interest	221.44	Subvention Nat. Math. Magazine.	$ 400.00
Increase in value of securities	208.82	Balance, Dec. 31, 1943	8,436.24
	$8,836.24		$8,836.24

VIII. Chauvenet Fund

Balance, Jan. 15, 1943	$ 614.40		
Interest	16.19		
Increase in value of securities	15.26	Balance, Dec. 31, 1943	$ 645.85
	$ 645.85		$ 645.85

IX. Life Membership Fund

Balance, Jan. 15, 1943	$ 824.30	Transferred to Current Fund	$ 88.96
Interest	21.71	Balance, liability as of Dec. 31, 1943	777.52
Increase in value of securities	20.47		
	$ 866.48		$ 866.48

X. General Invested Fund

Balance, Jan. 15, 1943	$20,438.19		
Transferred from Current Fund	262.80		
Transferred from Current Fund	270.86		
Increase in value of securities	507.74	Balance, Dec. 31, 1943	$21,479.59
	$21,479.59		$21,479.59

XI. Total Funds of the Association on December 31, 1943

Current Fund (checking account)		$ 2,262.47
Savings Account, Cleveland Trust Company		1,721.97
Savings Account, Oberlin Savings Bank		811.20
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 7,016.73	
Chace Fund	8,974.07	
Houck Fund	8,436.24	
Chauvenet Fund	645.85	
Life Membership Fund	777.52	
General Fund	21,479.59	47,330.00
		$52,125.64

W. B. Carver, *Secretary-Treasurer*

THE TWENTY-NINTH ANNUAL MEETING OF THE KANSAS SECTION

A joint meeting of the Kansas Section of the Mathematical Association of America and the Kansas Association of Teachers of Mathematics was held at the University of Kansas, Lawrence, Kansas, on Saturday, April 10, 1943. Professor C. F. Lewis, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was fifty-six, including the following twenty-five members of the Association: R. W. Babcock, Wealthy Babcock, W. D. Bemmels, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, Paul Eberhart, W. H. Garrett, Edison Greer, J. R. Hanna, W. C. Janes, H. E. Jordan, C. F. Lewis, Thirza A. Mossman, G. B. Price, C. B. Read, J. A. G. Shirk, D. T. Sigley, G. W. Smith, E. B. Stouffer, W. T. Stratton, Gilbert Ulmer, E. B. Wedel, J. J. Wheeler, Ferna E. Wrestler.

The following officers were elected for the coming year: Chairman, Paul Eberhart, Washburn Municipal University of Topeka; Vice-Chairman, Edison Greer, Beech Aircraft Corporation, Wichita; Secretary, Anna Marm, Bethany College. The executive committee was requested to select the time and place for the next meeting.

Professor Gilbert Ulmer, chairman of the committee on the fourth mathematics placement test, reported on the tests given at the beginning of the 1942–1943 school year to entering students in most of the colleges and junior colleges in the state. After some discussion, the committee was asked to continue the test work for the year 1943–1944.

The following papers were presented:

1. *A correspondence refresher course for mathematics teachers*, by Professor Florence L. Black, University of Kansas.

The speaker gave a discussion of the refresher course for mathematics teachers given by the University of Kansas Bureau of Correspondence Study under the sponsorship of the Federal Government.

2. *Mathematics in war*, by Professor R. W. Babcock, Kansas State College.

3. *Mathematics in air navigation*, by Professor Paul Eberhart, Washburn Municipal University.

It was pointed out by Professor Eberhart that very little mathematics is used in the actual practice of air navigation, but that a considerable amount of elementary mathematics is desirable in the training of the navigator. He stated that the most important topics in mathematics for the navigator are simple arithmetic, significant figures, reading of tables, reading of graphs, and the slide rule. For an actual understanding of celestial navigation, solid geometry and spherical trigonometry are needed. In practice, mathematical results are obtained by means of tables and mechanical computers.

4. *Cryptography-secret writing*, by Professor G. W. Smith, University of Kansas.

Professor Smith pointed out that ciphers are usually of three main types, namely concealment ciphers, transposition ciphers, and substitution ciphers. Several slides were shown illustrating each of these types. Certain methods used in deciphering were illustrated. The speaker showed how the theory of finite fields is used to construct ciphers which might well be considered "unbreakable."

5. *Adjustments in mathematics to the impact of war*, by Professor G. B. Price, University of Kansas.

Professor Price's paper was published in this MONTHLY, vol. 59, 1943, pp. 31–34.

ANNA MARM, *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The eighteenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 27, 1943. Professor J. E. Davis, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was forty-one, including the following twenty-seven members of the Association: E. F. Allen, A. A. Bennett, H. W. Brinkmann, W. B. Campbell, P. A. Caris, J. W. Clawson, H. B. Curry, J. E. Davis, F. L. Dennis, Margaret C. Eide, Benjamin Epstein, C. E. Heilman, R. T. Luginbuhl, F. L. Manning, A. E. Meder, Jr., Lillian Moore, Richard Morris, W. R. Murray, C. A. Nelson, C. W. Pflaum, C. J. Rees, J. B. Rosser, L. L. Smail, G. L. Walker, A. D. Wallace, Anna Pell Wheeler, P. M. Whitman.

At the business meeting the following officers were elected for the next year: Chairman, Anna Pell Wheeler, Bryn Mawr College; Secretary, P. M. Whitman, University of Pennsylvania; Program Committee, H. W. Brinkmann, Swarthmore College (Chairman), Tomlinson Fort, Lehigh University, and F. L. Manning, Ursinus College. It was voted to hold the next meeting at the University of Pennsylvania, Philadelphia, Pa., on the Saturday following Thanksgiving, 1944. The date was made subject to change if necessary.

The following papers were presented:

1. *Fixed point theorems*, by Professor A. D. Wallace, University of Pennsylvania.

The Brouwer fixed point theorem was proved and applied to the theory of matrices.

2. *Some modern viewpoints on euclidian geometry*, by Major A. A. Bennett, Aberdeen Proving Ground.

The topic arose from discussion of an elementary Hermetian geometry of the general simplex. Special aspects concerned: (1) the use of analytic as con-

trasted with synthetic methods; (2) the use of a fundamental discontinuous complex field, extended by adjunction to form the scalar field; (3) the use of a finite but large dimensionality, special subspaces yielding subgeometries; (4) extension to imaginary elements by use of a fundamental Hermetian rather than quadratic form; (5) the use of particles as weighted points, and of other weighted figures; (6) the use not only of Hermetian hyperspheres (shells), but also of analogous algebraic-geometric elements to which no points are incident; (7) The use of a unique particle A at infinity; (8) the use of a basic fixed simplex from which, and A, all constructions are made; (9) the incidental existence of a pure inversive Hermetian geometry of many dimensions; (10) the covariant and contravariant aspects of the geometric elements. Simple applications were made to pencils of shells, and to some elementary metric theorems.

3. *Distance sets*, by Professor J. C. Oxtoby, Bryn Mawr College, introduced by Dr. Whitman.

The speaker considered the following question: if two finite sub-sets A and B of an abelian group have the same set to differences, when can it be inferred that A is congruent to B, or to $-B$ by a group translation? A Theorem of S. Piccard and some recent results of A. L. Patterson bearing on this problem were discussed. Patterson has shown that the problem is intimately connected with the question of the unique determination of crystal structure by means of x-ray analysis.

4. *Photographic astrometry*, by Professor Peter van de Kamp, Swarthmore College, introduced by Professor Dresden.

Photographic astrometry is the study of the accurate positions (*i.e.*, directions) of star images on photographs. This paper was limited to long focus photography, and particular attention was given to studies made from photographs taken in the focal plane of the Sproul visual refractor at Swarthmore College. By a linear transformation, the measurements on all plates of a certain star field are reduced to the same origin, scale, and orientation. With suitable precautions, relative positions are obtained with an accuracy of 0.01 second of arc.

With the aid of many slides, it was pointed out that the study of proper motion in the mass leads to results for solar motion and mean parallax; the study of individual proper motions in multiple systems and clusters leads to dynamical information about these systems; the study of orbital motion of double stars leads to information about stellar masses; perturbations in the motions of double and single stars indicate the presence of companion stars inaccessible to direct observation.

5. *Transcendentality of certain continued fractions*, by Professor G. C. Webber, University of Delaware, introduced by Professor Rees.

Professor Webber discussed the question of transcendence of certain classes of continued fraction expansions. Using the Siegel theorem that $(b_0, b_1, b_2, \cdots)$ is transcendental if the b_i form an arithmetic progression of order one, he showed

that $(b_0, a_1, a_2, \cdots, a_k, b_1, a_1, \cdots, a_k, b_2, \cdots)$ is also transcendental. Likewise, $(a, b, a+d, b+e, a+2d, b+2e, \cdots)$ is transcendental if a, d, and c are positive integers, c or d being even, and $e=c^2d$, $b=c^2a+(c^2d/2)$. The Gelfond theorem, that e^z is transcendental if $x\neq 0$ is algebraic, leads to the transcendence of $(1, 3u^2, 5, 7u^2, \cdots)$, and $(1, 3u^2-1, 1, 4, 1, 7u^2-1, 1, 8, \cdots)$ if u^2 is a positive integer.

6. *On the many-valued logics*, by Professor J. B. Rosser, National Defense Research Committee.

The usual mode of reasoning decrees that any proposition must have one of the two truth values, truth or falsehood. It is possible to allow many truth values, but then one must use different methods of reasoning. One such method was outlined by the speaker.

P. M. WHITMAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS, Normal, Ill., May 12–13, 1944
INDIANA, Indianapolis, November 10, 1944
IOWA, Cedar Rapids, April 15, 1944
KANSAS, Topeka, April 15, 1944
KENTUCKY, Lexington, April 29, 1944
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN, Ann Arbor, March 18, 1944
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1944
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

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